3.2 Numerical Methods for Antenna Analysis

The sinusoidal current model for a dipole antenna is convenient because antenna parameters can be derived in analytical form. Analytical results are useful for antenna synthesis and for gaining insight into the physical behavior of the many types of dipole-like antennas that are in use. Due to radiation of energy from the current mode on the antenna arms, however, the true current distribution on a dipole is not exactly sinusoidal. To obtain a better approximation, we can use a numerical algorithm to solve for the current on a dipole antenna.

The major categories of numerical methods used in antenna analysis and design are the finite difference time domain method (FDTD) and the finite element method (FEM), which transform the differential equations of electromagnetics into difference equations, and integral equation methods, which transform the differential equations of electromagnetics through the use of a Green's function into integral equations that can be solved using the method of moments (MoM).

We will consider a very simple implementation of MoM for a one-dimensional integral equation. The 1D MoM is one of the oldest and easiest numerical methods to implement. Many more sophisticated algorithms and software packages are available today, but the 1D case illustrates the basic concepts of numerical analysis and can actually be used to solve fairly sophisticated electromagnetics problems. The 1D integral equations are also referred to as thin-wire integral equations.

3.2.1 Pocklington's Integral Equation

The first step in obtaining a numerical method based on MoM is to derive an integral equation from the equations of electromagnetics. An integral equation is a relationship that involves an integral of an unknown quantity, where the value of the integral is known. The unknown quantity is typically the current distribution on an antenna or other radiating or scattering object. The integral is typically a convolution type integral, where the integrand consists of a bivariate kernel function or Green's function and the unknown current distribution. Once we have derived the integral equation, we will then transform it using the MoM technique into a matrix equation that can be solved using a computational algorithm.

To derive Pocklington's integral equation, we begin with the magnetic vector potential radiated by a z-directed dipole. Since the direction of the vector potential is the same as that of the current, $\overline{A}$ will only have a z component, in which case (2.62) becomes

$$E_z = -j\omega A_z - \frac{j}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial z^2} \tag{3.14}$$

Using the radiation integral (2.60) for the magnetic vector potential, this becomes

$$\int d\overline{r}' J_z(\overline{r}')\left[\frac{\partial^2}{\partial z^2} + k^2\right]\frac{e^{-jkR}}{4\pi R} = j\omega\epsilon E_z(\overline{r}) \tag{3.15}$$

where $R = |\overline{r} - \overline{r}'|$. On the surface of the conductor, the electric field boundary condition requires that

$$E_z + E_z^{\mathrm{inc}} = 0 \tag{3.16}$$

where E_z^{inc} is the incident electric field from the source at the feed gap that excites the antenna. This leads to the integral equation

$$\int d\overline{r}' J_z(\overline{r}')\left[\frac{\partial^2}{\partial z^2} + k^2\right]\frac{e^{-jkR}}{4\pi R} = -j\omega\epsilon E_z^{\mathrm{inc}}(\overline{r}) \tag{3.17}$$

We will now assume that the wire is thin, so that the current distribution does not vary significantly azimuthally around the wire. The current density can then be approximated as

$$J_z(\overline{r}) = \frac{I_z(z)}{2\pi a}\delta(\rho - a) \tag{3.18}$$

where a is the wire radius. Substituting this into (3.17) leads to

$$\begin{aligned} -j\omega\epsilon E_z^{\text{inc}}(\overline{r}) &= \int_0^\infty \int_0^{2\pi} \int_{-l/2}^{l/2} \frac{I_z(z')}{2\pi a}\delta(\rho'-a)\left[\frac{\partial^2}{\partial z^2}+k^2\right]\frac{e^{-jkR}}{4\pi R}\rho'\,d\rho'\,d\phi'\,dz' \\ &\simeq \int_{-l/2}^{l/2} I_z(z')\left[\frac{\partial^2}{\partial z^2}+k^2\right]\frac{e^{-jkR}}{4\pi R}\,dz' \end{aligned}$$

where

$$R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{3.19}$$

$$\simeq \sqrt{a^2+(z-z')^2} \tag{3.20}$$

Working the derivatives in this expression provides the final form of Pocklington's integral equation,

$$\int_{-l/2}^{l/2} I_z(z')\left[(1+jkR)(2R^2-3a^2)+(kaR)^2\right]\frac{e^{-jkR}}{4\pi R^5}\,dz' = -j\omega\epsilon E_z^{\text{inc}}(z) \tag{3.21}$$

One difficulty with this integral equation is the strong singularity of the kernel due to the R^{-5} term. This singularity makes it difficult to work the integrals used in the method of moments to transform the integral operator into a matrix. To avoid dealing with this singularity, alternate integral formulations are available.

3.2.2 Hallén's Integral Equation

Another 1D integral equation can be derived from the Helmholtz equation for A_z. Neglecting the variation of A_z in the x-y plane, the Helmholtz equation reduces to

$$\frac{\partial^2}{\partial z^2}A_z + k^2 A_z = 0 \tag{3.22}$$

on the surface of the conductor away from the driving source. In this equation, A_z represents the total field including the driving source, rather than only the radiated field as in the previous derivation. This differential equation has the solution

$$A_z(z) = -j\sqrt{\mu\epsilon}[A\cos(kz) + B\sin(k|z|)] \tag{3.23}$$

where we have enforced the symmetry of the problem about $z = 0$. From the Lorenz gauge, we have that

$$\nabla\cdot\overline{A} = -j\omega\mu\epsilon\phi$$

where ϕ is the electric potential. The gradient of $\overline{A} = A_z\hat{z}$ is

$$\frac{\partial A_z}{\partial z} = \begin{cases} -j\sqrt{\mu\epsilon}k[-A\sin(kz)+B\cos(k|z|)] & z>0 \\ -j\sqrt{\mu\epsilon}k[-A\sin(kz)-B\cos(k|z|)] & z<0 \end{cases}$$

We now assume that a phasor voltage V_i is impressed at $z = 0$ over an infinitesimal feed gap. This is called a delta-gap source. Due to the source, the electric potential must jump by V_i at $z = 0$, so that

$$\begin{aligned} -j\omega\mu\epsilon V_i &= \left[\left.\frac{\partial A_z}{\partial z}\right|_{z>0} - \left.\frac{\partial A_z}{\partial z}\right|_{z<0}\right] \\ &= -j\sqrt{\mu\epsilon}k2B \end{aligned}$$

from which we find that $B = V_i/2$. The radiation integral for $\overline{A}$ gives

$$A_z(z) \simeq \mu \int_{-l/2}^{l/2} I(z') \frac{e^{-jkR}}{4\pi R}\, dz' \tag{3.24}$$

Combining these expressions leads to the integral equation

$$\int_{-l/2}^{l/2} I(z') \frac{e^{-jkR}}{4\pi R}\, dz' = -\frac{j}{\eta}[A\cos(kz) + \frac{V_i}{2}\sin(k|z|)] \tag{3.25}$$

which has a less singular kernel than Pocklington's integral equation, and is therefore easier to integrate when applying the method of moments. The price we pay for the simpler kernel is the presence of an additional unknown coefficient that must be solved for along with the unknown current on the wire.

3.2.3 Method of Moments

In order to solve either of these integral equations for the current on the dipole, we can use a numerical method that provides an approximate solution for the current. The method of moments allows us to transform an integral equation into a matrix equation or linear system of equations. The linear system can be readily solved to obtain samples or weights that give the unknown current in the integral equation. The basic approach to transforming an integral equation into a linear system is the method of moments.

We expand the current distribution as a finite linear combination of known sources or expansion functions, so that

$$I(z) \simeq \sum_{n=1}^{N} c_n f_n(z) \tag{3.26}$$

where c_n are unknown coefficients and $f_n(z)$ are a set of known basis functions. Using this in the integral term of Hallén's integral equation leads to

$$\int_{-l/2}^{l/2} \sum_{n=1}^{N} c_n f_n(z') \frac{e^{-jkR}}{4\pi R}\, dz' = \sum_{n=1}^{N} c_n \int f_n(z') \frac{e^{jkR}}{4\pi R}\, dz' \tag{3.27}$$

The right and left-hand sides of (3.25) depend on z. We need to discretize the z dependence by projecting both sides onto a set of M testing functions $t_m(z)$:

$$\int t_m\, [\text{LHS}] = \int t_m\, [\text{RHS}] \tag{3.28}$$

This leads to

$$\sum_{n=1}^{N} c_n \underbrace{\left[\int\int t_m(z) f_n(z') \frac{e^{-jkR}}{4\pi R}\, dz\, dz'\right]}_{Z_{mn}} = -\frac{j}{\eta}\int t_m(z)\left[A\cos(kz) + \frac{V_i}{2}\sin(k|z|)\right] dz \tag{3.29}$$

This is a set of M linear equations with $N+1$ unknowns (c_n and A). We have identified the elements of the matrix as Z_{mn}. Because the current unknown has units of Amps and the right hand side has units of Volts, the units of the matrix elements are impedance (Ohms). Typically, we choose M so that the resulting linear system has the same number of equations and unknowns and the resulting matrix is square.

In order to proceed further, we need to make choices for the expansion and testing functions. Since the current on the dipole is symmetric, we only need unknown coefficients c_n for the current on one side of the dipole. Let z_n be a set of points along the dipole spaced Δz apart, so that

$$z_n = (n - 1/2)\Delta z, \quad n = 1, 2, \ldots, N, \quad \Delta z = \frac{l/2}{N+1} \tag{3.30}$$

We will choose the expansion and testing functions to be given by

$$f_n(z) = \begin{cases} 1 & z_n - \Delta z/2 \le z \le z_n + \Delta z/2 \\ 0 & \text{otherwise} \end{cases} \tag{3.31}$$

$$t_m(z) = \delta(z - z_m) \tag{3.32}$$

The expansion functions are pulse functions, and the testing functions are delta functions. The moment matrix elements become

$$Z_{mn} = \int_{z_n-\Delta z/2}^{z_n+\Delta z/2} \frac{e^{-jkR}}{4\pi R}\, dz', \quad R = \sqrt{a^2 + (z_m - z')^2} \tag{3.33}$$

The integral can be approximated using the midpoint rule, so that the value of the integral is approximated by the length of the interval of integration Δz multiplied by a sample of the integrand at the center z_n of the region of integration. This leads to

$$Z_{mn} \simeq \frac{e^{-jkR_{mn}}}{4\pi R_{mn}}\Delta z, \quad R_{mn} = \sqrt{a^2 + (z_m - z_n)^2} \tag{3.34}$$

for the elements of the moment matrix $\mathbf{Z}$.

The matrix equation in (3.29) provides N linear equations. Since there is one additional unknown, A, we need one more equation. To obtain this equation, we will use the fact that the current at the wire ends is zero, so that $I(\pm l/2) = 0$ and we do not need an unknown coefficient c_{N+1} for the pulse function at the end of the wire. Even though we will not use an expansion function at z_{N+1}, we will still include a testing function at z_{N+1}, which will provide another equation and lead to a square linear system (i.e., $M = N + 1$). Equation (3.29) can be written in matrix form as

$$\sum_{n=1}^{N} (Z_{mn} + Z_{mn}^-)c_n + \frac{j}{\eta} A\cos(kz_m) = -\frac{jV_i}{2\eta}\sin(k|z_m|) \tag{3.35}$$

The matrix with elements Z_{mn} is called the moment matrix. Z_{mn}^- is defined similarly to Z_{mn} in (3.29) but with $R_{mn} = \sqrt{a^2 + (z_m + z_n)^2}$, which accounts for the currents on the bottom half of the dipole. In array form,

$$\left[\begin{array}{c:c} Z_{mn} + Z_{mn}^- & \frac{j}{\eta}\cos(kz_m) \end{array}\right] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \\ A \end{bmatrix} = \begin{bmatrix} -\frac{jV_i}{2\eta}\sin(kz_m) \end{bmatrix}_{m=N+1}^{m=1} \tag{3.36}$$

This linear system can be solved for the unknown coefficients c_n which provides samples of the current distribution on the antenna.

With the current distribution, we can compute the input impedance and radiation pattern. The input impedance is

$$Z_{\text{in}} = \frac{V_i}{I(0)} = \frac{V_i}{c_1} \tag{3.37}$$

where c_1 is the current unknown located adjacent to the feed gap. To find the radiation pattern, we evaluate the radiation integral numerically using the midpoint integration rule. For a z-directed current, the radiation integral is

$$\begin{aligned} E_\theta(r,\theta,\phi) &= jk\eta\frac{e^{-jkr}}{4\pi r}\sin\theta\int_{-l/2}^{l/2} I(z')e^{jkz'\cos\theta}\,dz' \\ &\simeq jk\eta\frac{e^{-jkr}}{4\pi r}\sin\theta\int_{-l/2}^{l/2}\sum_n c_n f_n(z')e^{jkz'\cos\theta}\,dz' \\ &\simeq jk\eta\frac{e^{-jkr}}{4\pi r}\sin\theta\sum_{n=1}^{N} c_n(e^{jkz_n\cos\theta}+e^{-jkz_n\cos\theta})\Delta z \end{aligned}$$

This provides a way to obtain the far field of the dipole antenna from the current samples c_n.

Both Hallén's and Pocklington's integral equations break down and give poor results as the wire radius a becomes large. A typical value is $a = .005\,\lambda$, although useful results can be obtained with larger radii. Due to the ill-conditioning of the moment matrix for large values of N, the method of moments also leads to inaccurate results if too many basis functions are used. This is an undesirable property, because ideally a numerical method should converge to the exact value as the number of unknowns is increased. Because of this numerical instability, in modern software packages 1D integral equations have largely been replaced by more sophisticated 3D integral equations or methods based on finite differences or finite elements. With the 1D thin wire integral equation method developed in this section, good results can usually be obtained if $\Delta z \simeq 3a$.