8.3 Signal Correlation Matrices

So far, we have treated received and transmitted signals deterministically and have modeled antenna systems using current and voltage phasors that are independent of time. This implies that all received and transmitted waveforms are time-harmonic or single-frequency sinusoids. Since noise, interference, and even information-bearing signals can be modeled as random processes, we need to expand the treatment of phased array antennas to include the statistical properties of array output voltage waveforms.

A single noise signal can be represented as a random process with specified statistical properties. For array antennas, the noise signals at each element output can be correlated, so we will develop a formulation that quantifies the statistical properties of multiple correlated noise signals. This correlation matrix representation is a fundamental tool in array signal processing theory and has close connections to earlier results from antenna theory and network theory.

8.3.1 Random Processes and Baseband Representation of Noise

For an antenna that receives random noise or modulated signals, the complex baseband voltage phasor $v_m(t)$ at the mth output is a function of time. The real voltage signal before complex basebanding is related to the complex baseband phasor by

$$V_m(t) = \mathrm{Re}[v_m(t)e^{j\omega t}] \tag{8.15}$$

The bandwidth of $v_m(t)$ is set by a bandpass filter or other frequency-limited component in the system. If the passband has an ideal rect response shape, then the bandwidth of $v_m(t)$ is the bandwidth B of the passband. If the shape of the response is not ideal, then B can be obtained by a weighted integral of the response, and is referred to as the system noise equivalent bandwidth. Normally, B is much smaller than the system's microwave center frequency $\omega/(2\pi)$, so $v_m(t)$ can be thought of as the envelope of a random narrowband modulated sinusoid according to

$$V_m(t) = |v_m(t)|\cos[\omega t + \angle v_m(t)] \tag{8.16}$$

Since thermal noise and other contributions to $v_m(t)$ are nondeterministic, both the real signal $V_m(t)$ and the complex baseband representation $v_m(t)$ can be modeled as random processes. Instead of using realizations of the voltage signals in our analysis we will work with the statistics of the random processes. Typically, noise and signals of interest have no DC component and have zero mean. The most important statistical properties of the signal is the frequency content, as represented by the power spectral density, and the time-average power, which is propertional to the variance of the random process.

8.3.2 Random Vectors and Signal Correlation Matrices

For array antennas, we must extend the concept of variance to multiple signals modeled by a vector of random processes or random vector for short. The outputs of a receiving array can be represented by the random vector $\mathbf{v}(t) = [v_1(t)\ v_2(t) \cdots v_N(t)]^T$. The individual signals or random process may not be statistically independent, in which case we say that the signals are correlated. The fundamental second order statistic of a vector of random processes is the covariance matrix. For zero mean random processes, covariance is equal to correlation (see Section 10.1.2). As a matter of practical convenience, we will characterize array output signals using the correlation matrix instead of the covariance matrix.

The correlation matrix is defined by the expectation of the outer product of the random vector,

$$\mathbf{R_v} = \mathrm{E}[\mathbf{v}\mathbf{v}^H] \tag{8.17}$$

where $\mathrm{E}[\,\cdot\,]$ denotes expectation. Rigorously, the correlation matrix formulation holds for a temporally wide sense stationary system. In practice, the correlation matrix is computed using sample estimates according to

$$\hat{\mathbf{R}}_{\mathbf{v}} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{v}[n]\mathbf{v}[n]^H \tag{8.18}$$

where the hat denotes a sample estimated correlation matrix. A correlation matrix is always Hermitian and positive semidefinite. The diagonal elements of the correlation matrix gives the variance of each (zero mean) signal, and the off-diagonal elements quantify the degree of correlation of pairs of signals.

8.4 Signal and Noise Model for Receiving Arrays

For a receiving array, the array output voltages include signal and noise contributions according to

$$\mathbf{v} = \mathbf{v}_{\rm sig} + \underbrace{\mathbf{v}_{\rm ext} + \mathbf{v}_{\rm loss} + \mathbf{v}_{\rm rec}}_{\mathbf{v}_{\rm n}} + \mathbf{v}_{\rm int} \tag{8.19}$$

where $\mathbf{v}_{\rm sig}$ is the signal of interest, $\mathbf{v}_{\rm ext}$ is noise due to external sources, $\mathbf{v}_{\rm loss}$ is thermal noise due to losses in the array, $\mathbf{v}_{\rm rec}$ is noise due to the front end amplifiers and receivers, and $\mathbf{v}_{\rm int}$ is unwanted interference from transmitters other than the one radiating the signal of interest. In some cases, the external noise consists of thermal noise from ground, warm objects, and sky, but in high traffic frequency bands, interference from other transmitters is often dominant. In most cases, the signal, noise, and interference voltage waveforms are uncorrelated, so that

$$\mathrm{E}[\mathbf{v}_{\rm sig}\mathbf{v}_{\rm n}^H] = \mathrm{E}[\mathbf{v}_{\rm sig}\mathbf{v}_{\rm int}^H] = \mathrm{E}[\mathbf{v}_{\rm n}\mathbf{v}_{\rm int}^H] = \mathbf{0} \tag{8.20}$$

where $\mathbf{0}$ is the zero matrix. With this assumption, the array output correlation matrix is

$$\mathbf{R}_{\mathbf{v}} = \mathbf{R}_{\rm sig} + \underbrace{\mathbf{R}_{\rm ext} + \mathbf{R}_{\rm loss} + \mathbf{R}_{\rm rec}}_{\mathbf{R}_{\rm n}} + \mathbf{R}_{\rm int} \tag{8.21}$$

where each term is the correlation matrix of one of the voltage contributions (e.g., $\mathbf{R}_{\rm sig} = \mathrm{E}[\mathbf{v}_{\rm sig}\mathbf{v}_{\rm sig}^H]$).

The correlation matrix formulation is closely related to time average power. If the output voltage for a formed phased array beam is applied to a load resistance R, then the time average power dissipated in the load is

$$\begin{aligned} P &= \tfrac{1}{2R}\mathrm{E}[|v_{\rm out}|^2] \\ &= \tfrac{1}{2R}\mathrm{E}[\mathbf{w}^H\mathbf{v}\mathbf{v}^H\mathbf{w}] \\ &= \tfrac{1}{2R}\mathbf{w}^H\mathrm{E}[\mathbf{v}\mathbf{v}^H]\mathbf{w} \\ &= \tfrac{1}{2R}\mathbf{w}^H\mathbf{R}_{\mathbf{v}}\mathbf{w} \end{aligned} \tag{8.22}$$

For an array that uses digital beamforming, the array output correlation matrix is computed in digital signal processing, so the leading factor has little importance and we often simply refer to the quadratic form $\mathbf{w}^H\mathbf{R}_{\mathbf{v}}\mathbf{w}$ as the beam output power. Using (8.21), the beam output power can be expanded into its various signal and noise contributions.

It is also convenient to define the correlation matrix of open circuit voltages at the array element terminals, which from (8.4) is related to the receiver output voltage correlation matrix by

$$\mathbf{R}_{\mathbf{v}} = \mathbf{Q}\mathbf{R}_{\mathbf{v}_{\rm oc}}\mathbf{Q}^H \tag{8.23}$$

With this relationship, the signal and noise contributions can be characterized by the open circuit voltage correlation matrices $\mathbf{R}_{\text{sig,oc}}$, $\mathbf{R}_{\text{ext,oc}}$, $\mathbf{R}_{\text{loss,oc}}$, and $\mathbf{R}_{\text{rec,oc}}$. Since the receiver noise is introduced after the array element terminal reference plane, $\mathbf{R}_{\text{rec,oc}}$ is an equivalent correlation matrix in the same sense as the equivalent noise concept introduced in Section 2.5.

The goal now is to develop expressions for each of the signal and noise correlation matrices in terms of the signal and noise environment and the array electrical characteristics as represented by the embedded element patterns and mutual impedance matrix. This treatment will uncover some fascinating connections between array radiation properties and the noise response.

8.4.1 Signal of Interest

For a single, stationary, time harmonic point source, the array output voltages are constant phasors, so the signal correlation matrix is

$$\mathbf{R}_{\text{sig}} = \mathrm{E}[\mathbf{v}_{\text{sig}}\mathbf{v}_{\text{sig}}^H] = \mathbf{v}_{\text{sig}}\mathbf{v}_{\text{sig}}^H \tag{8.24}$$

Since the field arriving at the array from a distant point source is approximately a plane wave, we can use (8.4) and (8.10) to express the signal correlation matrix in terms of the embedded element patterns as

$$\mathbf{R}_{\text{sig}} = c_2 S^{\text{sig}}\mathbf{Q}\underbrace{\mathbf{E}_p(\overline{r})\mathbf{E}_p^H(\overline{r})}_{\mathbf{R}_{\text{sig,oc}}}\mathbf{Q}^H \tag{8.25}$$

where $c_2 = 2\eta|c_1|^2$, S^{sig} is the incident power density of the signal at the location of the array, and $\overline{r}$ is a point such that $\hat{r}$ is in the direction Ω. This shows that the correlation matrix for a single point source is a rank one matrix. It can be seen that $\mathbf{R}_{\text{sig,oc}}$ is proportional to the matrix $\mathbf{B}_p$ defined in (7.17). If interference is produced by point radiators, then $\mathbf{R}_{\text{int}}$ is a sum over terms of the same form as (8.24).

8.4.2 External Noise

External thermal noise can be modeled in terms of a scene brightness temperature distribution $T(\Omega)$. We wish to obtain an expression for $\mathbf{R}_{\text{ext}}$ in terms of $T(\Omega)$. $T(\Omega)$ may represent the physical temperature of the ground and objects around the array, or for an environment with many interferers $T(\Omega)$ can represent an effective interference brightness temperature. If $T(\Omega)$ is constant, the noise environment is spatially isotropic. This particular noise distribution will have special importance in Chapter 8.

Because the noise consists of a distribution of randomly polarized incoming plane waves from all angles rather than a single plane wave, (8.9) must be modified by integrating over all angles and both polarizations must be included to obtain the received open circuit voltage noise at the mth element. To avoid having to repeat the transfer matrix $\mathbf{Q}$ in every expression, it is convenient here to use open circuit voltages. The open circuit voltage signal due to external noise at the mth element terminals is

$$v_{\text{ext,oc},m}(t) = c_1 \int \overline{E}^{\text{ext}}(\Omega)\cdot \overline{E}_m(\overline{r})\, d\Omega$$

where c_1 is defined in (8.11) and $\overline{E}^{\text{ext}}(\Omega)$ is the incident electric field from the thermal noise source at spherical angle of arrival Ω from the point of view of the array. The voltage correlation matrix has elements given by

$$
\begin{aligned}
R_{\text{ext,oc},mn} &= \mathrm{E}\left[|c_1|^2 \oint \overline{E}^{\text{ext}}(\Omega)\cdot\overline{E}_m(\overline{r})\,d\Omega \oint \overline{E}^{\text{ext}*}(\Omega')\cdot\overline{E}_n^*(\overline{r})\,d\Omega'\right] \\
&= |c_1|^2 \oint\oint \mathrm{E}\left[\overline{E}^{\text{ext}}(\Omega)\cdot\overline{E}_m(\overline{r})\overline{E}^{\text{ext}*}(\Omega')\cdot\overline{E}_n^*(\overline{r})\right] d\Omega\,d\Omega' \\
&= |c_1|^2 \oint\oint \mathrm{E}\left[(E_\theta^{\text{ext}}(\Omega)E_{m,\theta} + E_\phi^{\text{ext}}(\Omega)E_{m,\phi})(E_\theta^{\text{ext}}(\Omega')E_{n,\theta} + E_\phi^{\text{ext}}(\Omega')E_{n,\phi})^*\right] d\Omega\,d\Omega'
\end{aligned}
$$

From the physics of blackbody radiation, the noise field correlations are

$$
\begin{aligned}
\mathrm{E}\left[E_\theta^{\text{ext}}(\Omega){E^{\text{ext}}}_\theta^*(\Omega')\right] = \mathrm{E}\left[E_\phi^{\text{ext}}(\Omega){E^{\text{ext}}}_\phi^*(\Omega')\right] &= \frac{2\eta k_{\mathrm{B}}T(\Omega)B}{\lambda^2}\delta(\Omega-\Omega') \\
\mathrm{E}\left[E_\theta^{\text{ext}}(\Omega){E^{\text{ext}}}_\phi^*(\Omega')\right] &= 0
\end{aligned}
$$

The delta function implies that the thermal noise waveforms radiated by two different sources are uncorrelated. Using these expressions and evaluating one of the integrals over spherical angle leads to

$$
\begin{aligned}
R_{\text{ext,oc},mn} &= |c_1|^2\frac{2\eta k_{\mathrm{B}}B}{\lambda^2}\oint T(\Omega)\overline{E}_m(\overline{r})\cdot\overline{E}_n^*(\overline{r})\,d\Omega \\
&= \frac{8k_{\mathrm{B}}B}{\eta|I_0|^2}\oint T(\Omega)\overline{E}_m(\overline{r})\cdot\overline{E}_n^*(\overline{r})\,r^2 d\Omega \qquad (8.26)
\end{aligned}
$$

If we define a brightness temperature weighted overlap integral matrix with elements given by

$$
A_{T(\Omega),mn} = \frac{1}{2\eta}\oint T(\Omega)\overline{E}_m(\overline{r})\cdot\overline{E}_n^*(\overline{r})\,r^2 d\Omega \qquad (8.27)
$$

then the external noise correlation matrix becomes

$$
\mathbf{R}_{\text{ext,oc}} = \tfrac{1}{|I_0|^2}16k_{\mathrm{B}}B\mathbf{A}_{T(\Omega)} \qquad (8.28)
$$

Using the transfer matrix $\mathbf{Q}$, the correlation matrix is transformed to

$$
\mathbf{R}_{\text{ext}} = \tfrac{1}{|I_0|^2}16k_{\mathrm{B}}B\mathbf{Q}\mathbf{A}_{T(\Omega)}\mathbf{Q}^H \qquad (8.29)
$$

when referred to receiver output voltages.

Isotropic external noise response.

A case of particular interest is an isotropic external brightness temperature distribution $T(\Omega) = T_{\text{iso}}$. For this thermal noise distribution, (8.28) becomes

$$
\mathbf{R}_{\text{ext,iso,oc}} = \tfrac{1}{|I_0|^2}16k_{\mathrm{B}}T_{\text{iso}}B\mathbf{A} \qquad (8.30)
$$

where $\mathbf{A}$ is the pattern overlap matrix defined in (7.11). In view of the relationship (7.46), this can be written as

$$
\mathbf{R}_{\text{ext,iso,oc}} = 8k_{\mathrm{B}}T_{\text{iso}}B\mathbf{R}_{\text{rad}} \qquad (8.31)
$$

where $\mathbf{R}_{\text{rad}}$ is defined in (7.47) to be $\mathrm{Re}[\mathbf{Z}_{\mathrm{A}}] - \mathbf{R}_{\mathrm{A,loss}}$. For a lossless array, these results imply that the pattern overlap matrix $\mathbf{A}$, the array mutual resistance matrix $\mathrm{Re}[\mathbf{Z}_{\mathrm{A}}]$, and the open circuit isotropic noise voltage correlation matrix $\mathbf{R}_{\text{ext,iso,oc}}$ are all identical up to a scale factor.

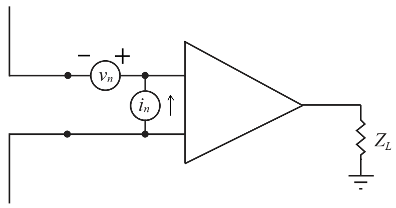

Figure 8.2: Amplifier noise equivalent circuit model.

8.4.3 Loss Noise

Thermal noise due to losses in the array can be inferred from Twiss's theorem [7]. If the array is in thermal equilibrium with an isotropic noise environment (i.e., the physical temperature T_p equal to the external brightness temperature T_iso), then the open circuit thermal noise correlation matrix at the antenna terminals is

$$\mathbf{R}_\mathrm{t,oc} = 8k_\mathrm{B}T_\mathrm{iso}B\mathrm{Re}[\mathbf{Z}_\mathrm{A}] \tag{8.32}$$

This correlation matrix includes contributions both from the external isotropic brightness temperature distribution and ohmic losses in the array. By subtracting $\mathbf{R}_\mathrm{t,oc}$ from $\mathbf{R}_\mathrm{ext,iso,oc}$, we find that the loss noise contribution is

$$\mathbf{R}_\mathrm{loss,oc} = 8k_\mathrm{B}T_\mathrm{p}B\mathbf{R}_\mathrm{A,loss} \tag{8.33}$$

This expression may be notationally confusing, because the voltage correlation matrix on the left side and the mutual loss resistance matrix on the right side are both denoted by a matrix symbol $\mathbf{R}$. But in view of the close connection between noise correlation matrices and mutual resistances, perhaps this chance collision of the nomenclature from network theory and array signal processing is not such a bad thing.

8.4.4 Receiver Noise

For an active array with amplifiers directly attached to each element, receiver noise is dominated by noise from the front end amplifiers. For a microwave amplifier, the noise contributed by the amplifier depends strongly on the impedance of the source attached to the amplifier input. Amplifier noise can be modeled with voltage and current noise sources at the input port of an ideal, noiseless amplifier as in Figure 8.2. The amplifier noise is then determined by the source impedance together with the noise parameters of the amplifier, which are the amplifier noise correlation admittance Y_c, the voltage noise RMS density $\bar{v}_{n,R}$ (V/$\sqrt{\mathrm{Hz}}$), and the current noise RMS density $\bar{i}_{n,R}$ (A/$\sqrt{\mathrm{Hz}}$). The correlation admittance Y_c measures the degree of correlation of the voltage and current noise sources, and is defined by

$$i_{n,R} = Y_\mathrm{c}v_{n,R} + i_{u,R} \tag{8.34}$$

where $i_{u,R}$ and $v_{n,R}$ are uncorrelated. Another useful parameter is the amplifier noise resistance,

$$R_\mathrm{N} = \frac{\bar{v}_{n,R}^2}{4k_\mathrm{B}T_0} \tag{8.35}$$

where $T_0 = 290\,\mathrm{K}$. The equivalent noise temperature is strongly dependent on the impedance of the source that drives the amplifier input. The lowest possible value of the equivalent noise temperature is

$$T_\mathrm{min} = \frac{\bar{v}_{n,R}\bar{i}_{n,R}}{2k_\mathrm{B}}\left(\sqrt{1-c_i^2} + c_r\right) \tag{8.36}$$

where the voltage and current noise correlation coefficient $c = c_r + jc_i$ is defined by $Y_\mathrm{c} = c\bar{i}_{n,R}/\bar{v}_{n,R}$. The optimal source admittance that minimizes the noise contributed by the amplifier is

$$Y_\mathrm{opt} = \frac{T_\mathrm{min}}{2R_\mathrm{N}T_0} - Y_\mathrm{c} \tag{8.37}$$

For any other source admittance, the equivalent amplifier noise temperature is larger than T_min. The optimal source impedance parameter is not in general the same as the amplifier input impedance, but to maximize the power gain of the amplifier and minimize the effect of noise introduced by later components in the receiver chain, amplifiers are typically designed so that the optimal source impedance is close to the input impedance.

Measuring all of the amplifier noise parameters (Y_c, $\bar{i}_{n,R}$, and $\bar{i}_{n,R}$) requires specialized equipment. For many commercial amplifiers, the optimal source impedance is close to $50\,\Omega$, and only the minimum noise temperature T_min, or equivalently the amplifier noise figure, are given in the product specifications. Amplifier noise can also be characterized in terms of forward and reverse noise wave amplitudes at the amplifier input or output port or other quantities that can be measured. Transformations between various sets of noise parameters are found in J. Engberg and T. Larsen, *Noise Theory of Linear and Nonlinear Circuits* [New York: Wiley, 1995].

For an array with multiple amplifiers, if we arrange the noise parameters for the N amplifiers into diagonal matrices $\mathbf{Y}_c$, $\mathbf{V}_{n,R}$, and $\mathbf{I}_{n,R}$, then network theory can be used to show that the amplifier noise correlation matrix is

$$\mathbf{R}_\text{rec,oc} = 2B\left(\mathbf{V}_{n,R}^2 + \mathbf{Z}_A\mathbf{Y}_c\mathbf{V}_{n,R}^2 + \mathbf{V}_{n,R}^2\mathbf{Y}_c^H\mathbf{Z}_A^H + \mathbf{Z}_A\mathbf{I}_{n,R}^2\mathbf{Z}_A^H\right) \tag{8.38}$$

in terms of the amplifier noise parameters and the array mutual impedance matrix.

This result includes the effect of mutual coupling in the array through the nondiagonal matrix $\mathbf{Z}_A$. We have neglected signal coupling between the amplifiers themselves by making $\mathbf{Y}_c$, $\mathbf{V}_{n,R}$, and $\mathbf{I}_{n,R}$ diagonal matrices, but this is normally a good assumption in microwave systems. The noise emitted by each amplifier is assumed to be uncoupled with that of other amplifiers, but the reverse noise at the input port of one amplifier enters into the array and couples to other elements, and enters the forward signal paths of the other array elements, which makes $\mathbf{R}_\mathrm{rec}$ a nondiagonal matrix.

8.5 Fundamental Noise Theorem for Phased Arrays

The results in the previous section on the noise response of an array receiver create an important and interesting connection between the radiation properties, noise response, and network parameters of an array antenna. Combining (8.31) and (8.32) leads to

$$\mathbf{R}_\text{t,oc} = \tfrac{1}{|I_0|^2}16k_\mathrm{B}T_\mathrm{iso}B\mathbf{A} + 8k_\mathrm{B}T_\mathrm{p}B\mathbf{R}_\text{A,loss} \tag{8.39}$$

where $\mathbf{R}_\text{A,loss}$ is the portion of the array mutual impedance matrix $\mathbf{Z}_A$ that is due to ohmic and dielectric losses in the array elements and is given by (7.47). This result holds under assumption that the array elements are reciprocal (although system components after the array elements may not be reciprocal). We will refer to this result as the fundamental noise theorem of array receivers.

The left hand side of (8.39) is the correlation matrix of the noise voltages that appear at the element ports of an array when loaded with open circuits and the array is in an isotropic noise environment at temperature T_iso and with the array at temperature T_p. This correlation matrix consists of two contributions, one from the external environment and the other from losses in the array elements. The external contribution can be given in terms of the array element pattern overlap integral matrix, and the loss term is given in terms of the part of the array mutual impedance matrix caused by losses in the array elements. This result combines all aspects of array theory—the signal correlation matrix formulation, antenna quantities through the pattern overlap matrix, and network theory via the mutual impedances.

8.6 Array Gain (SNR Gain)

In addition to the receiving pattern directivity defined in Section 8.2.1, another way to measure the performance of a receive array is the improvement in signal to noise ratio relative to a reference antenna. Since gain scale factors in the array signal paths affect signal and noise in the same way, the SNR at the output of a beamformer is independent of gain scale factors common to each element signal path, and SNR can be used to characterize the ability of an array to receive a particular signal in terms of the ratio of the signal power to the received noise at the beamformer output.

Because the SNR at the beamformer output depends on the intensity of the incident field, it is convenient to normalize the output SNR by the SNR obtained with a reference antenna for the same incident field and noise environment, to obtain a figure of merit that is intrinsic to the array. This is the array gain,

$$G_a = \frac{\mathrm{SNR}_{\mathrm{out}}}{\mathrm{SNR}_{\mathrm{ref}}} \tag{8.40}$$

where $\mathrm{SNR}_{\mathrm{out}}$ is the signal to noise ratio at the beamformer output and $\mathrm{SNR}_{\mathrm{ref}}$ is the signal to noise ratio at the output of a reference antenna, which can be chosen to be one of the array elements in isolation or an isotropic antenna. For obvious reasons, this quantity is also be called SNR gain. The name "array gain" for this quantity is used by the signal processing community, even though it is generally different from antenna gain or directivity. We will see shortly, however, that under certain conditions array gain is actually equal to directivity.

From the definition, it can be seen that the array gain not only depends on the response of the array in the signal direction, but also on the sources of noise in the system. In general, the noise consists of thermal radiation from warm objects around the antenna, thermal noise from the antenna elements, and receiver noise due to the amplifiers and other components connected to the output port of the array elements. If these noise sources change, then the array gain changes. Therefore, the array gain depends on the characteristics of the various sources of system noise.

When designing an array it is sometimes convenient to use figures of merit that are intrinsic to the array, so it is common to compute the array gain assuming a generic type of noise model. Two possible noise models that are natural choices include an isotropic external thermal noise distribution and independent, identically distributed noise at each element, or spatially white noise. We will show that the array gain with the former noise model (isotropic noise gain) is identical to the partial directivity of the array.

8.6.1 Isotropic Noise Gain

With certain choices for the reference antenna and noise model, it can be shown that array gain and directivity are equal. We will choose an isotropic antenna as the reference, and assume that the noise consists only of isotropic thermal radiation arriving from all angles around the array. This noise model neglects receiver noise, ohmic losses in the antenna elements, and other noise sources. Since the noise model consists of spatially isotropic thermal noise, the resulting array gain can be called the isotropic noise gain.

If all other noise sources except for noise due to an external isotropic brightness temperature distribution are neglected, then the output SNR due to a plane wave signal arriving from the spherical angle Ω is

$$\mathrm{SNR}_{\mathrm{out}} = \frac{\mathbf{w}^H \mathbf{R}_{\mathrm{sig}}(\Omega)\mathbf{w}}{\mathbf{w}^H \mathbf{R}_{\mathrm{ext,iso}}\mathbf{w}} \tag{8.41}$$

$$= \frac{\lambda^2 r^2 S^{\mathrm{sig}}}{k_{\mathrm{B}} T_{\mathrm{iso}} B} \frac{\mathbf{w}_{\mathrm{oc}}^H \mathbf{B}_p(\Omega)\mathbf{w}_{\mathrm{oc}}}{\mathbf{w}_{\mathrm{oc}}^H \mathbf{A}\mathbf{w}_{\mathrm{oc}}} \tag{8.42}$$

The reference SNR is given by the SNR at the terminals of an isotropic antenna. Using (2.118), the signal power is

$$P_{\mathrm{sig}} = S^{\mathrm{sig}} \frac{\lambda^2}{4\pi} \tag{8.43}$$

From (2.136), the noise power due to an isotropic external brightness temperature distribution is

$$P_{\mathrm{noise}} = k_{\mathrm{B}} T_{\mathrm{iso}} B \tag{8.44}$$

The SNR at the terminals of the isotropic antenna is

$$\mathrm{SNR}_{\mathrm{ref}} = \frac{\lambda^2 S^{\mathrm{sig}}}{4\pi k_{\mathrm{B}} T B} \tag{8.45}$$

Dividing (8.42) by (8.45) leads to the array gain for isotropic noise,

$$G_a(\Omega) = \frac{4\pi r^2 \mathbf{w}_{\mathrm{oc}}^H \mathbf{B}_p(\Omega)\mathbf{w}_{\mathrm{oc}}}{\mathbf{w}_{\mathrm{oc}}^H \mathbf{A}\mathbf{w}_{\mathrm{oc}}} \tag{8.46}$$

This is identical to the partial directivity (7.19). If the polarization $\hat{p}$ of the incident field is aligned with the antenna polarizations of the array elements, then the distinction between partial directivity and directivity disappears and the isotropic noise gain in (8.46) is identical to the directivity.

8.6.2 White Noise Gain

Another common choice for the noise model is to simply approximate the noise as independent, identically distributed noise sources for each array element, or spatially white noise. This is often used in the array signal processing literature when the details of the array system are not important in the analysis. In this approximation, the noise correlation matrix is

$$\mathbf{R}_{\mathrm{n,oc}} = \sigma_{\mathrm{n}}^2 \mathbf{I} \tag{8.47}$$

and the noise power is

$$P_{\mathrm{noise}} = \sum_{m,n} w_{\mathrm{oc},m}^* w_{\mathrm{oc},n} \sigma_{\mathrm{n}}^2 = \sigma_{\mathrm{n}}^2 \mathbf{w}_{\mathrm{oc}}^H \mathbf{w}_{\mathrm{oc}} \tag{8.48}$$

where σ_{n}^2 is the variance of the open circuit noise voltage at one receiver output. The output SNR is

$$\mathrm{SNR}_{\mathrm{out}} = \frac{c_2 S^{\mathrm{sig}} \mathbf{w}_{\mathrm{oc}}^H \mathbf{B}_p(\overline{r})\mathbf{w}_{\mathrm{oc}}}{\sigma_{\mathrm{n}}^2 \mathbf{w}_{\mathrm{oc}}^H \mathbf{w}_{\mathrm{oc}}} \tag{8.49}$$

If we take the first array element as the reference, then

$$\mathrm{SNR}_{\mathrm{ref}} = \frac{c_2 S^{\mathrm{sig}} B_{p,11}(\Omega)}{\sigma_{\mathrm{n}}^2} \tag{8.50}$$

The white noise gain is

$$G_a = \frac{1}{B_{p,11}} \frac{\mathbf{w}_{\text{oc}}^H \mathbf{B}_p(\overline{r}) \mathbf{w}_{\text{oc}}}{\mathbf{w}_{\text{oc}}^H \mathbf{w}_{\text{oc}}} \tag{8.51}$$

If we instead use an isotropic antenna as the reference, the white noise gain becomes

$$G_a = \frac{4\pi r^2}{P_{\text{el}}} \frac{\mathbf{w}_{\text{oc}}^H \mathbf{B}_p(\overline{r}) \mathbf{w}_{\text{oc}}}{\mathbf{w}_{\text{oc}}^H \mathbf{w}_{\text{oc}}} \tag{8.52}$$

This result can also be arrived at from the partial directivity by assuming that the array elements are identical and far enough apart that the off-diagonal pattern overlap matrix elements are zero, so that $\mathbf{A} = P_{\text{el}}\mathbf{I}$.

8.7 Antenna Terms for Active Receiving Arrays

The signal processing figure of merit of SNR gain or array gain can be easily applied to any receiver system, no matter how complicated. All that is required is an identifiable output voltage or power and a measure of the signal and noise contributions at the output, so that the SNR at the output of the receiver system can be quantified. The directivity of a receiver is also well defined and can be measured (See Section 8.2).

Other antenna terms, such as radiation efficiency or gain, are more challenging to apply to complex receiver systems. For arrays with active elements such as amplifiers or digitally beamformed arrays with analog to digital converters, the system is nonreciprocal and cannot be operated as transmitters. There is no obvious way to define an equivalent radiated power or measure the radiation efficiency. The standard for antenna terms now includes terms that address these limitations.

The 2013 update of the IEEE Standard for Definitions of Terms for Antennas [2] includes several new terms that clarify quantities like efficiency, gain, and noise temperature for active array antennas. The key to understanding gain, radiation efficiency, and noise temperature for complex receiving antenna systems is noise theory. We have already seen in Section 8.5 that for arrays of reciprocal antenna elements, no matter how complicated the system, there is a close connection between the fields radiated by a transmitting array and the noise received by the same array. The relevant antenna terms are

- Isotropic noise response
- Active antenna available gain
- Active antenna available power
- Receiving efficiency
- Noise matching efficiency
- Noise temperature
- Effective area

We will review each of these antenna terms and show how they resolve the issues associated with characterizing performance for array receiver systems.

8.7.1 Isotropic Noise Response

We have already seen the important special case of isotropic external brightness temperature distribution in Section 8.4.2. The isotropic noise response is similar to the response of an array when in an isotropic external brightness temperature distribution, but includes loss noise as well. Definitions of antenna quantities for active antenna systems all rely on the receiver's isotropic noise response, so this figures heavily into the treatment of array receivers and will be treated in greater depth in this section.

The IEEE standard definition of the isotropic noise response of a receiver is as follows:

> **isotropic noise response.** For a receiving active array antenna, the noise power at the output of a formed beam with a noiseless receiver when in an environment with brightness temperature distribution that is independent of direction and in thermal equilibrium with the antenna.

For a receiving array, the isotropic noise response can be modeled theoretically using the correlation matrix formulation. The array output noise voltage correlation matrix given by (8.21) in general consists of the contributions

$$\mathbf{R}_{\mathbf{v}} = \mathbf{R}_{\text{sig}} + \underbrace{\mathbf{R}_{\text{ext}} + \mathbf{R}_{\text{loss}}}_{\mathbf{R}_{\text{t}}} + \mathbf{R}_{\text{rec}} + \mathbf{R}_{\text{int}} \tag{8.53}$$

The isotropic noise response can be found from the external noise and loss noise correlation matrix terms, which we denote as the thermal noise correlation matrix $\mathbf{R}_{\text{t}}$. For an arbitrary external environment temperature distribution and antenna temperature, the thermal noise correlation matrix is given by (8.39), the fundamental noise theorem of array receivers.

The isotropic noise response is obtained under the conditions that the external environment has a constant brightness temperature distribution T_{iso} and the array is thermal equilibrium with the environment, so that the physical temperature is also T_{iso}. The isotropic thermal noise correlation matrix in this scenario becomes

$$\mathbf{R}_{\text{t,iso}} = \mathbf{R}_{\text{ext,iso}}(T_{\text{iso}}) + \mathbf{R}_{\text{loss}}(T_{\text{iso}}) \tag{8.54}$$

where the argument is a reminder that the external environment and the antenna array are at temperature T_{iso}.

By Twiss's theorem (8.32), the isotropic thermal noise correlation matrix when referenced to open circuit voltages at the array element ports is proportional to the array mutual resistance matrix Re$[\mathbf{Z}_{\text{A}}]$. This demonstrates that the open circuit isotropic noise correlation matrix is intrinsic to the antenna array itself, and the dependence of $\mathbf{R}_{\text{t,iso}}$ on the electronics after the array elements is through the system transfer matrix $\mathbf{Q}$ defined in (8.4).

From the isotropic noise correlation matrix, the beam isotropic noise response can be expressed as

$$P_{\text{t,iso}} = \mathbf{w}^H \mathbf{R}_{\text{t,iso}} \mathbf{w} \tag{8.55}$$

This provides a way to compute the isotropic noise response if the isotropic noise correlation matrix is known, and gives us a way to understand the isotropic noise response in terms of the various noise contributions to the array output voltage signals.

The isotropic noise response can be measured in several ways. The array mutual impedances can be measured using a network analyzer. By Twiss's theorem, this provides the open circuit isotropic noise correlation matrix. The system transfer matrix $\mathbf{Q}$ from the antenna port open circuit reference plane to the array output voltage reference plane must then be measured or modeled to obtain the isotropic noise correlation matrix at the array outputs after electronics. The beamformer weights can then be applied to find the isotropic noise response (8.55).

Another approach is to create an isotropic brightness temperature distribution and measure the isotropic noise response directly. For high frequency antennas that are small in size, it is possible to create an artificial black body using microwave absorber cooled with liquid nitrogen or other type of cryogenic system to maintain constant temperature. As long as the absorber fills the majority of the angular field of view of the antenna where the gain is substantial, it appears to be an isotropic brightness temperature distribution from the perspective of the antenna.

For large array systems, the cool microwave sky provides an approximate isotropic noise field. The atmosphere, Milky Way galaxy, and the cosmic background radiation together create a brightness temperature of about 4-5 K at L-band frequencies. A ground shield must be used to limit radiation from the much warmer ground, terrain features, trees, or buildings in the deep sidelobes of the antenna system under test (AUT).

With either of these measurement setups, the isotropic noise response of an array must be separated from receiver noise. This can be done using the Y-factor method. The Y-factor method is normally used to measure the noise figure of microwave devices, with a connectorized noise source that can be switched between two noise levels. To measure the isotropic noise response of an antenna, a spatial noise field with two temperatures is required, rather than a connectorized noise field.

The AUT is illuminated with an isotropic noise field at two different temperatures, T_{hot} and T_{cold}. These are referred to as hot and cold loads. This is done using absorber at two different temperatures, or by using the night sky as the cold load and microwave absorber at ambient temperature. The beam output power is measured twice, with the cold load and hot load. The measured output powers are

$$P_{\text{cold}} = P_{\text{ext,iso}}\frac{T_{\text{cold}}}{T_{\text{iso}}} + P_{\text{loss}} + P_{\text{rec}} \tag{8.56a}$$

$$P_{\text{hot}} = P_{\text{ext,iso}}\frac{T_{\text{hot}}}{T_{\text{iso}}} + P_{\text{loss}} + P_{\text{rec}} \tag{8.56b}$$

where $P_{\text{ext,iso}}$ is similar to (8.55) but is external noise due to the isotropic noise field but does not include noise due to antenna losses. The antenna in the measurement setups is normally not at thermal equilibrium with the external noise field, so P_{loss} represents loss noise with the antenna at its physical temperature, not T_{cold} or T_{hot}. This means that the loss contribution to the isotropic noise response cannot be directly measured, and must be neglected, measured another way, or estimated using simulations. The scale factors $T_{\text{cold}}/T_{\text{iso}}$ and $T_{\text{hot}}/T_{\text{iso}}$ change the effective external brightness temperature in the first terms of (8.56) from T_{iso} to the hot and cold temperatures.

The ratio of the hot and cold output powers is referred to as the Y factor,

$$Y = \frac{P_{\text{hot}}}{P_{\text{cold}}} \tag{8.57}$$

Subtracting the measured output powers with hot and cold loads and inserting the Y factor leads to

$$P_{\text{ext,iso}} = P_{\text{cold}}(Y-1)\frac{T_{\text{iso}}}{T_{\text{hot}} - T_{\text{cold}}} \tag{8.58}$$

We can also solve for the loss and receiver noise contributions,

$$P_{\text{loss}} + P_{\text{rec}} = P_{\text{cold}}\frac{YT_{\text{cold}} - T_{\text{hot}}}{T_{\text{hot}} - T_{\text{cold}}} \tag{8.59}$$

Expressions very similar to the above formulas can be derived for the external isotropic noise, loss noise, and receiver noise array output voltage correlation matrices. If the array output voltages can be correlated before combining in the beamformer, then this same measurement setup can be used to obtain the external isotropic noise correlation matrix.

8.7.2 Active Antenna Available Gain

The power output of an active array receiver includes essentially arbitrary gain scale factors. These include amplifier gain, loss in connecting transmission lines, conversion loss in mixers, transfer functions of analog to digital converters, and a common scale factor in the array beamformer weights. This makes the absolute level of the signal at the output of a beamforming array almost meaningless. The level must be scaled to be within the dynamic range of components in the analog and digital portions of the signal chain for each channel, but other than that, the output level is meaningless.

To remove the voltage gain scale factors in the array system and separate out the antenna gain of the array, we use the isotropic noise response of the active array to define an active antenna available gain:

> **active antenna available gain.** For a receiving active array antenna, the ratio of the isotropic noise response to the available power at the terminals of any passive antenna over the same bandwidth and in the same isotropic noise environment.

The available power at the terminals of any passive antenna in an environment with brightness temperature $T_{\rm iso}$ over a bandwidth B is given by $k_{\rm b}T_{\rm iso}B$. The active antenna available gain is therefore

$$G_{\rm rec}^{\rm av} = \frac{P_{\rm t,iso}}{k_{\rm b}T_{\rm iso}B} \tag{8.60}$$

This quantity can be used to scale the beam output power for an active array so that it represents the available power at the terminals of an equivalent passive antenna. This is the active antenna available power.

8.7.3 Active Antenna Available Power

With the active antenna available gain, the beam output power can be scaled to remove system gain factors so that represents the available power at the terminals of an equivalent passive antenna with the same receiving pattern. The active antenna available power is defined to be

> **active antenna available power.** For a receiving active array antenna, the power at the output of a formed beam divided by the active antenna available gain.

If we consider only the contribution to the antenna output power due to the signal of interest, the active antenna available signal power is

$$P_{\rm sig}^{\rm av} = \frac{P_{\rm sig}}{G_{\rm rec}^{\rm av}} \tag{8.61}$$

where $P_{\rm sig} = \mathbf{w}^H\mathbf{R}_{\rm sig}\mathbf{w}$.

The relationship developed earlier between the isotropic noise correlation matrix and the array mutual resistance matrix can be used to show that the maximum active antenna available power over all possible beamformer weights is equal to the available power at the array element terminals. For a single point source, $\mathbf{R}_{\rm sig} = \mathbf{Q}\mathbf{v}_{\rm sig,oc}\mathbf{v}_{\rm sig,oc}^H\mathbf{Q}^H$, where $\mathbf{v}_{\rm sig,oc}$ is a vector of the open circuit voltages induced by the signal of interest at the array element terminals. With (8.32), the beam equivalent available power due to the signal of interest is

$$P_{\rm sig}^{\rm av} = \frac{\mathbf{w}_{\rm oc}^H\mathbf{v}_{\rm sig,oc}\mathbf{v}_{\rm sig,oc}^H\mathbf{w}_{\rm oc}}{8\mathbf{w}_{\rm oc}^H\,{\rm Re}[\mathbf{Z}_{\rm A}]\mathbf{w}_{\rm oc}} \tag{8.62}$$

where $\mathbf{w}_{\rm oc} = \mathbf{Q}^H\mathbf{w}$. Following the treatment in Section 7.6.2, the beamformer weight vector that maximizes this quadratic form can be shown to be $\mathbf{w}_{\rm oc} = {\rm Re}[\mathbf{Z}_{\rm A}]^{-1}\mathbf{v}_{\rm sig,oc}$. The resulting active antenna available signal power with these beamformer weights is

$$P_{\rm sig}^{\rm av,max} = \tfrac{1}{8}\mathbf{v}_{\rm sig,oc}^H\,{\rm Re}[\mathbf{Z}_{\rm A}]^{-1}\mathbf{v}_{\rm sig,oc} \tag{8.63}$$

For a reciprocal array ($\mathbf{Z}_\mathrm{A} = \mathbf{Z}_\mathrm{A}^T$), this expression can be recognized as the signal power that would be delivered by the array to a conjugate matched multiport load network with mutual impedance matrix $\mathbf{Z}_\mathrm{L} = \mathbf{Z}_\mathrm{A}^H$ attached to the array element terminals. This result provides an intuitive physical meaning for active antenna available power. The maximum active antenna available power is the actual available power at the array element ports, considering the array as a multiport network source.

8.7.4 Receiving Efficiency

For transmitting antennas, the radiation efficiency is the ratio of the power radiated by the antenna to the power accepted by the antenna. This term is difficult to apply to complex, active antennas that include gain in multiple signal paths. There is no way to operate such a system as a transmitter to measure accepted power or the radiated power. Still, the ohmic and dielectric losses that absorb some of the power accepted by a transmitting antenna do have a detrimental effect on receiving antennas. The losses attenuate the received signal of interest, but that can be compensated for simply by adding gain to the signal path after the antenna. More importantly, the losses add thermal noise, which reduces the SNR at the receiver output. The IEEE standard definition for receiving efficiency measures this effect for active receiving antenna systems. For simple, passive antennas that operate as both receivers and transmitters, the radiation efficiency and receiving efficiency are equal.

> **receiving efficiency.** For a receiving active array antenna, the ratio of the isotropic noise response with noiseless antenna to the isotropic noise response, per unit bandwidth and at a specified frequency.
>
> NOTE—Equivalent to radiation efficiency for a passive, reciprocal antenna.

From this definition, the receiving efficiency is

$$\eta_\mathrm{rec} = \frac{P_\mathrm{ext,iso}}{P_\mathrm{t,iso}} = \frac{P_\mathrm{ext,iso}}{P_\mathrm{ext,iso} + P_\mathrm{loss}} \tag{8.64}$$

where the beam output noise power due to antenna losses, $P_\mathrm{loss} = \mathbf{w}^H \mathbf{R}_\mathrm{loss} \mathbf{w}$, is measured under the condition that the antenna and isotropic environment are in thermal equilibrium, so that the physical temperature T_p of the antenna is equal to T_iso. This ratio measures the increase in noise at the receiver output due to antenna losses. If the antenna elements are lossless, $P_\mathrm{loss} = 0$, and the receiving efficiency is unity. For highly lossy elements, the noise added by losses is large in comparison to the external noise contribution at the receiver output from the isotropic thermal environment, and the receiving efficiency is close to zero.

It is interesting to compare receiving and radiation efficiency for a simple passive antenna. For a passive antenna with conjugate matched load and loss factor L, from (2.139) the isotropic noise response is

$$P_\mathrm{t,iso} = k_\mathrm{B} B \frac{T_\mathrm{a}}{L} + k_\mathrm{B} B \frac{(L-1)T_\mathrm{p}}{L} \tag{8.65}$$

with the external and antenna physical temperatures equal, so that $T_\mathrm{a} = T_\mathrm{p} = T_\mathrm{iso}$. The first term is the power at the antenna output port due to the external thermal noise, and the second term is the noise contributed by antenna losses. The receiving efficiency is therefore

$$\begin{aligned}\eta_\mathrm{rec} &= \frac{k_\mathrm{B} B \frac{T_\mathrm{iso}}{L}}{k_\mathrm{B} B \frac{T_\mathrm{iso}}{L} + k_\mathrm{B} B \frac{(L-1)T_\mathrm{iso}}{L}} \\ &= \frac{1}{L}\end{aligned} \tag{8.66}$$

As discussed in Section 2.5.4, the loss factor of a lossy antenna is the inverse of the radiation efficiency, so that $L = 1/\eta_{\rm rad}$, which shows that $\eta_{\rm rec} = \eta_{\rm rad}$, and as expected the radiation and receiving efficiencies are equal for passive, reciprocal antennas.

We can develop a connection between receiving efficiency and radiation efficiency for array antennas using the overlap integral formulation. Radiation efficiency is defined for a transmitter as the ratio of radiated power to accepted power, which in the overlap integral formulation (7.51) is

$$\eta_{\rm rad} = \frac{2}{|I_0|^2}\frac{\mathbf{w}^H\mathbf{A}\mathbf{w}}{\mathbf{w}^H{\rm Re}[\mathbf{Z}_{\rm A}]\mathbf{w}} \tag{8.67}$$

Assuming that the array elements are reciprocal, then inserting (8.30) and (8.32) to convert from a power formulation to a noise formulation shows that (8.67) for the radiation efficiency can be expressed as

$$\eta_{\rm rad} = \frac{\mathbf{w}^H\mathbf{R}_{\rm ext,iso,oc}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_{\rm t,iso,oc}\mathbf{w}} \tag{8.68}$$

If the array is operated as a receiver instead of a transmitter, and the receiver open circuit referenced beamformer weights $\mathbf{w}_{\rm oc}$ are equal to the transmit beamformer weights $\mathbf{w}$, then the numerator and denominator of (8.68) are equal to the received external isotropic noise power $P_{\rm ext,iso}$ and total isotropic noise power $P_{\rm t,iso}$, respectively. By comparison with definition (8.64) of receiving efficiency the radiation and receiving efficiencies are therefore equal if the transmit and receive beamformer weight vectors are equal.

Based on the arguments in Section 8.1.1, this equivalence between radiation and receiving efficiencies holds for arrays of passive, reciprocal elements, even if components in the signal paths after the receiving array elements are nonreciprocal (e.g., amplifiers, mixers, etc.). If the antenna elements themselves include nonreciprocal materials, the radiation and receiving efficiency of the array may be different.

8.7.5 Active Antenna Effective Area

The passive equivalent available power defined in (8.61) allows the effective area of an active antenna to be defined in the same way as for a passive antenna. This is included as the third note under the definition of effective area:

> **effective area (of an antenna) (in a given direction).** In a given direction, the ratio of the available power at the terminals of a receiving antenna to the power flux density of a plane wave incident on the antenna from that direction, the wave being polarization matched to the antenna. See: polarization match.
>
> NOTE 1—If the direction is not specified, the direction of maximum radiation intensity is implied.
>
> NOTE 2–The effective area of an antenna in a given direction is equal to the square of the operating wavelength times its gain in that direction divided by 4π.
>
> NOTE 3—For an active antenna, available power is the active antenna available power.

Based on this definition, the effective area of an active antenna is

$$A_{\rm e} = \frac{P_{\rm sig}^{\rm av}}{S_{\rm sig}} \tag{8.69}$$

where the signal of interest is a plane wave with power flux density $S_{\rm sig}$ and polarization matched to the beam such that $A_{\rm e}$ is at a maximum. For an arbitrarily polarized incident field, partial gain and polarization efficiency can be defined according to the usual conventions. By inserting expressions for the power and active antenna available power for a beamforming array, the effective are can be written as

$$A_{\text{eff}} = \frac{P_{\text{sig,a}}}{S^{\text{sig}}} = \frac{k_{\text{B}} T_{\text{iso}} B}{S^{\text{sig}}} \frac{\mathbf{w}^H \mathbf{R}_{\text{sig}} \mathbf{w}}{\mathbf{w}^H \mathbf{R}_{\text{t}} \mathbf{w}} \tag{8.70}$$

in terms of the array output voltage covariance matrices. For a lossless array, $\mathbf{R}_{\text{t}} = \mathbf{R}_{\text{ext,iso}}$, in which case the ratio of the signal response to the isotropic noise response in this expression is the same as the ratio in the SNR gain with isotropic noise model given by (8.41), which shows that the antenna gain of an active array is the same as the isotropic noise gain.

We can use reciprocity to show that (8.69) is consistent with the standard definition for effective receiving area. For the array used as a transmitter with input currents at each element port arranged into the column vector $\mathbf{i}$, the partial directivity is given by (7.19) as

$$D_p(\Omega) = \frac{4\pi r^2 \mathbf{i}^H \mathbf{B}_p(\overline{r}) \mathbf{i}}{\mathbf{i}^H \mathbf{A} \mathbf{i}} \tag{8.71}$$

By inserting (8.25) and (8.30) for the received signal noise correlation matrix and isotropic noise correlation matrix, the partial directivity can be expressed for the receiving array as

$$D_p = \frac{4\pi k_{\text{B}} T_{\text{iso}} B}{\lambda^2 S^{\text{sig}}} \frac{\mathbf{w}_{\text{oc}}^H \mathbf{R}_{\text{sig,oc}} \mathbf{w}_{\text{oc}}}{\mathbf{w}_{\text{oc}}^H \mathbf{R}_{\text{ext,iso,oc}} \mathbf{w}_{\text{oc}}} \tag{8.72}$$

where the open circuit referenced beamformer weights are $\mathbf{w}_{\text{oc}} = \mathbf{i}^*$, or equal to the complex conjugates of the input currents used to excite the array as a transmitter. Comparing (8.72) and (8.70) and using the definition of receiving efficiency shows that

$$A_{\text{eff}} = \frac{\lambda^2}{4\pi} \eta_{\text{rec}} D_p \tag{8.73}$$

which is equivalent to the classical result (2.118) for lossless passive antennas. This derivation shows the consistency of the active antenna effective area with the antenna terms for passive antennas.

As expected for an array receiver, the effective area of the array depends on the beamformer coefficients. By analogy with (7.38), the beamformer weights that maximize the effective receiving area are given by

$$\mathbf{w}_{\text{oc}} = \mathbf{R}_{\text{t}}^{-1} \mathbf{E}_p \sim \text{Re}[\mathbf{Z}_{\text{A}}]^{-1} \mathbf{E}_p \tag{8.74}$$

Since antenna figures of merit are independent of a common scale factor in the beamformer coefficients, both expressions on the right of (8.74) represent essentially the same formed beam. For a lossless array, because the overlap matrix and the mutual resistance matrix are proportional, this reduces to (7.38). We can also determine beamformer weights that maximize sensitivity in a similar way. Since this beamformer depends on the external noise environment, however, it is not intrinsic to the array itself, but depends on external stimuli and is said to be data-dependent. This type of statistically optimal beamformer will be considered further in Chapter 13.

Comparing the results in Section 8.2.1, Section 8.6.1, and this section shows that for a reciprocal array of antenna elements, if the array excitation currents $\mathbf{i}$ in the transmit case and the open circuit referenced beamformer combining coefficients $\mathbf{w}_{\text{oc}}^*$ in the receive case are equal, then the directivity, isotropic noise gain, and active antenna effective area are all essentially the same quantity. The connecting link between the transmit and receive cases is that in both operating modes, we model the array in terms of the same embedded element radiated field patterns. The embedded element patterns are used to get the radiated power density and radiated power in the transmit case, and the signal correlation matrix and isotropic noise correlation matrix in the receive case. The equivalence between measures of gain and directivity for the receive and transmit cases boils down to the fact that the integral one evaluates to obtain total radiated power in (7.11) is the same as the integral over received field arrival angle in the receiving pattern directivity (8.12) and the integral over angle in the external noise response (8.27). In the receiving pattern directivity, a plane wave from many angles is used to probe the antenna response over angle and create the integral, whereas in the isotropic noise gain and the active antenna effective area, thermal noise arriving from all angles effectively performs the same integration over the element patterns.

8.7.6 Active Antenna Noise Temperature

In Section 8.7.3, we defined an equivalent available power that represents the output power of an active antenna as if it were at the terminals of the array antenna elements. The full receiver system may be active and nonreciprocal, but we have assumed throughout this treatment that the antenna elements themselves are always passive and reciprocal. The value of having a meaningful equivalent passive available power for active antennas is that this provides a way to refer power at the antenna system output to the source at the input to the system. The source in an active antenna system is the array element multiport network. In the previous section, we considered the example of referring signal power to the array multiport network source. Referring noise power to the source is even more important, as this is fundamental to the definition of equivalent noise temperature. For an active antenna system, noise powers can be referred to equivalent source temperatures by applying the second note in the IEEE standard definition of noise temperature:

> **noise temperature of an antenna.** The temperature of a resistor having an available thermal noise power per unit bandwidth equal to that at the antenna's output at a specified frequency.
>
> NOTE 1—Noise temperature of an antenna depends on its coupling to all noise sources in its environment, as well as noise generated within the antenna.
>
> NOTE 2—For an active antenna, the temperature of an isotropic thermal noise environment such that the isotropic noise response is equal to the noise power at the antenna output per unit bandwidth at a specified frequency.

This definition can be applied to the entire system noise power, or it can be applied separately to individual noise contributions that one would like to measure, understand, or optimize in a design process. The definition of active antenna noise temperature implies that the equivalent system noise temperature including all noise contributions is

$$T_{\text{sys}} = T_{\text{iso}} \frac{P_{\text{n}}}{P_{\text{t,iso}}} \tag{8.75}$$

This is the temperature of the antenna and isotropic noise environment that would produce a power output at the antenna system output equal to the system noise power $P_{\text{n}} = \mathbf{w}^H \mathbf{R}_{\text{n}} \mathbf{w}$. Active antenna noise temperature can be related to active antenna available gain. Using (8.60) in (8.75) shows that

$$P_{\text{n}} = k_{\text{B}} B T_{\text{sys}} G_{\text{rec}}^{\text{av}} \tag{8.76}$$

Based on the definition (2.123) of equivalent temperature, this can be seen to be exactly the noise power we would expect at the output of a system with gain $G_{\text{rec}}^{\text{av}}$.

The definition of active antenna noise temperature can also be applied to individual components of the system noise due to external thermal sources, antenna losses, or receiver electronics. The beam equivalent receiver noise temperature, for example, is

$$T_{\text{rec}} = T_{\text{iso}} \frac{P_{\text{rec}}}{P_{\text{t,iso}}} \tag{8.77}$$

where $P_{\text{rec}} = \mathbf{w}^H \mathbf{R}_{\text{rec}} \mathbf{w}$. The equivalent noise temperature due to antenna losses is

$$T_{\text{loss}} = T_{\text{iso}} \frac{P_{\text{loss}}}{P_{\text{t,iso}}} \tag{8.78}$$

The noise power due to losses is the difference between the total thermal noise power $P_{\text{t,iso}}$ and the external noise power $P_{\text{ext,iso}}$, but scaled so that the effective temperature of the antenna is T_{p}, rather than T_{iso} as in the definition of the isotropic noise response. This implies that the loss noise is

$$P_{\text{loss}} = \frac{T_{\text{p}}}{T_{\text{iso}}}(P_{\text{t,iso}} - P_{\text{ext,iso}}) \tag{8.79}$$

Inserting this into (8.78) and using the definition of receiving efficiency leads to

$$T_{\text{loss}} = (1 - \eta_{\text{rec}})T_{\text{p}} \tag{8.80}$$

which is identical to the second term of (2.140) for a passive antenna.

By convention, the equivalent system noise temperature and receiver noise temperature are referenced to the antenna ports after antenna losses, whereas external noise sources are referenced to an antenna temperature before losses (i.e., "to the sky"). This is analogous to the difference between (2.136) for the external antenna noise temperature T_{a} of a passive antenna before losses and (2.140) for the antenna noise temperature T'_{a} after losses. The reference power for the external noise temperature does not include noise due to antenna losses, so the equivalent external noise temperature

$$T_{\text{ext}} = T_{\text{iso}} \frac{P_{\text{ext}}}{P_{\text{ext,iso}}} \tag{8.81}$$

has a slightly different form as compared to the system noise and receiver noise temperatures.

With these definitions for the receiver, loss, and external noise temperatures, it can be shown that the system noise temperature (8.75) can be expressed as

$$T_{\text{sys}} = \eta_{\text{rec}} T_{\text{ext}} + (1 - \eta_{\text{rec}})T_{\text{p}} + T_{\text{rec}} \tag{8.82}$$

where T_{p} is the physical temperature of the antenna, as before. This expression is identical in form to the system temperature of a single port antenna (2.142), except that we have labeled the antenna temperature due to external noise sources as T_{ext}. This is often denoted in the antenna literature as the antenna temperature T_{a}, as in (2.136). The first two terms of this expression represent the total thermal noise due to external sources and antenna losses, which was denoted in (2.140) as T'_{a}.

We can also define the receiver noise figure of an active antenna using the isotropic noise response. The noise figure of a microwave device was defined in Section 2.5 as the noise power at the output of the device divided by the noise power at the output if the device were noise-free. Noise figure in principle depends on the temperature of the input noise reference used to measure the noise increase due to a noisy system component, so we must choose a reference input noise level. For microwave components with transmission line ports, the reference input noise power level is $k_{\text{B}} T_0 B$. For active antennas, where the "input" is a spatial thermal noise distribution in the environment around the antenna, the angular distribution of the thermal noise must also be specified. The IEEE standard for the noise temperature of an active antenna implies that the reference angular distribution is isotropic. For an active array, the receiver noise figure is therefore

$$F = \frac{P_{\text{t,iso}} + P_{\text{rec}}}{P_{\text{t,iso}}} \tag{8.83}$$

This quantity measures only the noise increase due to receiver electronics, and does not include the noise increase due to antenna losses. Using (2.127) to convert this to an equivalent temperature shows that this yields the same result as (8.77), which shows the consistency of the IEEE standard antenna terms for active antennas with standard concepts from microwave systems theory.

8.7.7 Equivalent Receiver Noise and Noise Matching

The antenna terms we have considered so far deal mainly with the external signals received by the antenna (effective area) or noise caused by losses in the array elements (receiving efficiency). For an active antenna receiver, the system noise includes contributions from receiver electronics as well. From the treatment of Section 8.4.4, the noise added by a two-port network such as an amplifier connected to an antenna depends on the impedance of the source. For a single antenna, since the effective area is defined with respect to available power, the effective area or antenna gain are not influenced by the impedance of the amplifier or other components connected to the antenna. The decrease in sensitivity caused by the receiver electronics can be easily accounted for with the receiver noise figure. For an active antenna array, it is more difficult to separate the properties of the antenna from the receiver noise figure, because the receiver has multiple signal chains, and the array antenna output is not formed until after the receiver chains.

We have already seen in Section 8.7.6 how the active antenna available gain can be used to define a meaningful equivalent noise temperature for any noise contribution at the active antenna output. This can be applied to the receiver noise as well. For an active antenna system, using (8.75) the equivalent receiver noise temperature is

$$T_{\rm rec} = \frac{P_{\rm rec}}{P_{\rm t,iso}} \tag{8.84}$$

In terms of array output voltage correlation matrices, the receiver noise temperature becomes

$$T_{\rm rec} = T_{\rm iso}\frac{\mathbf{w}^H\mathbf{R}_{\rm rec}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_{\rm t}\mathbf{w}} \tag{8.85}$$

Using the results of Section 8.4.4 for the receiver noise correlation matrix, the receiver noise temperature can be expressed in terms of open circuit voltage equivalent quantities as

$$T_{\rm rec} = \frac{1}{4k_{\rm B}}\frac{\mathbf{w}_{\rm oc}^H\left(\mathbf{V}_{n,R}^2 + \mathbf{Z}_A\mathbf{Y}_c\mathbf{V}_{n,R}^2 + \mathbf{V}_{n,R}^2\mathbf{Y}_c^H\mathbf{Z}_A^H + \mathbf{Z}_A\mathbf{I}_{n,R}^2\mathbf{Z}_A^H\right)\mathbf{w}_{\rm oc}}{\mathbf{w}_{\rm oc}^H{\rm Re}[\mathbf{Z}_{\rm A}]\mathbf{w}_{\rm oc}} \tag{8.86}$$

where we have assumed that the front end amplifiers are the dominant contribution to the receiver noise and have neglected noise from receiver components after the front-end amplifiers.

The receiver noise temperature can also be expressed in terms of active impedances or active reflection coefficients looking from the input port of each amplifier into the array element ports. It can be shown that [8]

$$T_{\rm rec} = T_{\rm min} + T_0\frac{\sum_{m=1}^M |w_{{\rm oc},m}|^2 R_{{\rm N},m}|Z_{{\rm act},m}|^2|Y_{{\rm act},m} - Y_{{\rm opt},m}|^2}{\sum_{m=1}^M |w_{{\rm oc},m}|^2 R_{{\rm act},m}} \tag{8.87}$$

where $T_{\rm min}$ is the minimum amplifier noise temperature parameter for the front-end amplifiers, $R_{{\rm N},m}$ is the noise resistance parameter for the mth amplifier, $Y_{{\rm opt},m}$ is the optimal source impedance parameter for the mth amplifier, $R_{{\rm act},m} = {\rm Re}\,[Z_{{\rm act},m}]$, $Y_{{\rm act},m} = 1/Z_{{\rm act},m}$, and the receive array active impedances $Z_{{\rm act},m}$ are defined as in (7.53) using the open circuit referenced beamformer weights $\mathbf{w}_{\rm oc}$.

This expression is comparable to the classical result for amplifier noise with a single antenna in terms of the antenna impedance $Z_{\rm in} = R_{\rm in} + jX_{\rm in}$ or admittance $Y_{\rm in} = 1/Z_{\rm in}$ and the amplifier noise parameters $T_{\rm min}$, $R_{\rm N}$, and $Y_{\rm opt}$. For $M = 1$, (8.87) reduces to

$$T_{\rm rec} = T_{\rm min} + T_0\frac{R_{\rm N}|Z_{\rm in}|^2|Y_{\rm in} - Y_{\rm opt}|^2}{R_{\rm in}} \tag{8.88}$$

which is equivalent to the usual formula for the equivalent noise temperature of a two-port amplifier with the antenna as its driving source at the input port. This result uncovers another connection between transmit and receive arrays. For transmit arrays, active impedances control the amount of power accepted by each array element from its feeding source, whereas for receive arrays, active impedances control the equivalent front-end amplifier noise. In both cases, for array antennas it is active impedances, and not the self impedances $Z_{\mathrm{A},nn}$ or the isolated impedance Z_{in} that controls the matching performance of the system.

A subtle point for receiving arrays that is often misunderstood is the effect of a poor impedance match between an antenna port and its terminating front-end amplifier. To understand this issue, it is important to first understand the difference between impedance matching and noise matching.

For a transmitter, an impedance mismatch between the driving power amplifier and a transmitting antenna means that less power is accepted by the array from the driver amplifiers than the available power. This effect is not included in the antenna gain, but is included as a factor in the realized gain.

For a receiving array, what happens if the impedance looking into the antenna port is different from the optimal source impedance parameter of the amplifier? The effective receiving area of the antenna is not affected, because receiving area is defined for a conjugate matched load at the antenna port. Instead, the equivalent receiver noise increases. By convention in microwave systems analysis, noise powers at the output of a system are often referred to equivalent input noise powers. This is done for convenience, because minimizing the input-referred equivalent noise maximizes SNR at the system output.

Equivalent receiver noise includes two effects that tend to decrease SNR—first, the reduction in signal intensity due to the impedance mismatch, and second, the increase in noisiness of the amplifier due to a difference between the source impedance and the amplifier's optimal source impedance. If there is a poor match between the antenna and amplifier, then only a small amount of signal couples from the antenna output port to the amplifier input port, and the equivalent noise power referred to the source (the antenna) must be very large to produce a given noise power at the amplifier output.

Both of these effects of mismatches, the decrease in coupled signal and the increase in amplifier noisiness, are incorporated in the optimal source impedance parameter of the amplifier. The SNR at the amplifier output is mathematically independent of the amplifier's input impedance. What matters in determining the SNR is not the amplifier input impedance, but the amplifier optimal source impedance parameter. In practice, the optimal source impedance and the input impedance of the amplifier are connected by the physical properties of the transistor and other components in the amplifier, and in a well designed amplifier are close in value, and typically as close to $50\,\Omega$ as possible. Since the input impedance and the optimal source impedance can be different, we refer to the process of matching the antenna to the amplifier optimal source impedance for maximum SNR not as impedance matching, but as noise matching. In general, impedance matching maximizes signal power transfer, whereas noise matching maximizes SNR.

A good impedance match between the antenna and amplifier, while it does not directly affect SNR at the amplifier output, does produce a higher power level at the amplifier output, meaning a higher available gain. With a high available gain for the first stage amplifier, later amplifiers do not need as much gain to avoid reducing the SNR significantly.

To summarize, mismatches in a receiver system lead to reduced coupling of the signal of interest from the antenna to the front end amplifier, as well as increased added noise caused by the amplifier. The impact of these effects on SNR is reflected in the equivalent receiver noise.

To account for the effects of mismatch in receiver systems, some authors have defined a matching efficiency parameter to account for poor impedance matching, similar to the mismatch factor that appears in the realized gain used for transmitting antenna systems. Because noise and mismatch effects are closely intertwined in a receiver system, however, including mismatch effects in the equivalent receiver noise rather than antenna gain is the most natural approach. This convention has been adopted in the latest IEEE Standard for Antenna Terms under the term noise matching efficiency [2].

8.7.8 Noise Matching Efficiency

The quality of the noise match between antenna elements and amplifiers or other receiver electronics in an active antenna system is quantified by the noise matching efficiency. Noise matching efficiency is defined as follows:

> **noise matching efficiency.** For a receiving active array antenna, the ratio of the noise power contributed by receiver electronics at the output of a formed beam, with receivers impedance matched to the array elements for minimum receiver noise temperature, to the actual receiver electronics noise power at the formed beam output, per unit bandwidth and at a specified frequency.

If the minimum equivalent noise temperature of each LNA and receiver chain is $T_{\rm min}$, then the receiver noise temperature under perfect noise matching at each element port is equal to $T_{\rm min}$. From the definition, this means that the noise matching efficiency of an active antenna system is

$$\eta_{\rm n} = \frac{T_{\rm min}}{T_{\rm rec}} = \frac{T_{\rm min}}{T_{\rm iso}} \frac{P_{\rm t,iso}}{P_{\rm rec}} \tag{8.89}$$

The receivers and array element ports are ideally matched if the optimal source impedance parameter for each LNA is equal to the active impedance at the corresponding array element port [9, 8]. In this case, $T_{\rm rec} = T_{\rm min}$ and the noise matching efficiency is unity. When the amplifiers and receiver chains are not perfectly noise matched to the array at each port, the receiver noise temperature is greater than $T_{\rm min}$ and the noise matching efficiency is less than unity. Since active reflection coefficients depend on beamformer coefficients, the noise matching efficiency for an active array depends on the beam steering direction.

The system noise temperature (8.82) can be expressed in terms of the noise matching efficiency as

$$T_{\rm sys} = \eta_{\rm rec} T_{\rm ext} + (1 - \eta_{\rm rec}) T_{\rm p} + T_{\rm min}/\eta_{\rm n} \tag{8.90}$$

where $T_{\rm ext}$ is the equivalent external noise temperature and $T_{\rm p}$ is the physical temperature of the antenna array elements.

As discussed in the previous section, noise matching efficiency includes both the effects of poor signal coupling from the array elements to receiver electronics and the increased noisiness of the receivers as a two-port network connected to each array element. Other antenna parameters, such as gain and effective area, are defined with reference to conjugate matched loads and available power, and do not depend on the impedance match to the receiver electronics. All mismatch effects in an active receiver system are contained in the noise matching efficiency (8.89).

It is interesting to consider the case of very small receiver noise or very large external noise. In this case, an impedance mismatch between the array elements and receiver electronics means that very little signal is delivered to the receiver chains. Normally, if the signal level that couples into the front end amplifiers is small, then the noise added by the receiver electronics would drastically reduce the SNR, but if the receivers add very little noise in relation to the external noise, the SNR is unaffected. The SNR is dominated by the ratio of signal to external noise, which is independent of the impedance matching at the junction between the antenna elements and receiver chains, since the signal and external noise couple in the same way from the antenna elements to the receiver chains. In other words, in the case of low receiver noise, the impedance match does not affect SNR. There may be other reasons to have a good impedance match, such as reducing the number of amplifier stages needed to achieve a given signal level at the receiver output, but in terms of SNR, the impedance match only impacts SNR if the added receiver noise is significant.

8.8 Receiving Array Sensitivity

The antenna terms defined in the previous sections describe the performance of various aspects of an active array antenna. Aperture efficiency, system noise, receiving efficiency, and other parameters all combine to determine the overall sensitivity of the system. Using the beam equivalent system temperature (8.75) and the effective receiving area (8.70), the receiver sensitivity figure of merit (2.144) for an array is

$$\frac{A_{\mathrm{eff}}}{T_{\mathrm{sys}}} = \frac{k_{\mathrm{B}}B}{S^{\mathrm{sig}}}\frac{\mathbf{w}^H\mathbf{R}_{\mathrm{sig}}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_{\mathrm{n}}\mathbf{w}} \tag{8.91}$$

in units of m^2/K. Sensitivity expressed as G/T_{sys} can be defined similarly. The ratio of quadratic forms on the right-hand side is the SNR at the beam output, so it can be seeen that this expression is identical to (2.144).

Inserting the definitions of receiving efficiency, aperture efficiency, external noise temperature, and receiver noise temperature shows that the sensitivity can be expressed as

$$\frac{A_{\mathrm{eff}}}{T_{\mathrm{sys}}} = \frac{\eta_{\mathrm{rec}}\eta_{\mathrm{ap}}A_{\mathrm{p}}}{\eta_{\mathrm{rec}}T_{\mathrm{ext}} + (1-\eta_{\mathrm{rec}})T_{\mathrm{p}} + T_{\mathrm{min}}/\eta_{\mathrm{n}}} \tag{8.92}$$

If the antenna is a phased array feed with a reflector, then the external noise can be expanded into a combination of sky noise and spillover noise as was done in (5.32). This result shows that the antenna terms defined in this section can be combined into a system-level figure of merit in a consistent way.

For a complex active receiving array system, the design can be analyzed and optimized by considering each of the efficiency parameters in this expression and maximizing each of them by adjusting the array elements to maximize receiving efficiency and achieve a good impedance match to the receiver electronics. Since these efficiencies all depend on beamformer coefficients, the beamforming process must be considered together with the array element design, which means that optimizing highly sensitive active array receivers is a challenging and interesting engineering problem.