Chapter 6

Array Antennas

An array antenna is a group of antenna elements with excitations coordinated in some way to achieve desired properties for the combined radiation pattern. When designing an array antenna, we have control over the element types, orientations, locations, and excitation currents in the transmit case. The basic principle of array antennas is that the relative delays of the excitation signals at each element determines the radiation pattern of the array.

For a simple single-beam array antenna, the excitation currents can be controlled using a fixed circuit structure with a tree structure of power splitters, delay lines, and possibly attenuators. This is known as a passive or corporate fed array.

A more sophisticated array uses electronic control of the element excitations to control the radiation pattern dynamically to steer the main beam of the array radiation pattern. This is an active array or electronically steered beamforming array. For a narrowband system, the variable delays in the signal paths for each element required to steer the beam can be represented as phase shifts. If the element excitation amplitude and phase are both varied, the combination can be viewed as a complex voltage coefficient. The complex excitations for the array elements are often referred to as array weights. Electronically controlled phase shifters and possibly variable gain amplifiers can be used to generate the array weights for a beamforming array. A basic transmitting array block diagram is shown in Figure 6.1.

The next step in system complexity is an array that uses digital signal processing (DSP) and digital to analog converters for each array element to synthesize the array excitations. This approach offers total control over the array radiation pattern. These kinds of systems are referred to as digital beamforming (DBF) arrays, smart arrays, or adaptive arrays. The disadvantage is the expense of the DSP hardware, particularly for broadband arrays, which require high bit rate processing.

For receiving arrays, the same system architectures are available. A corporate feed structure can be used to combine the signals from each element into a single beam output signal with fixed delays. Electronically controlled phase shifters and variable gain amplifiers can be used in the signal paths before a power combiner. For an adaptive receiving array, the element outputs are sampled and processed using DSP. The latter approach allows multiple simultaneous beams to be formed, so that the beamforming array has multiple outputs, each with a different receiving pattern. By reciprocity, the transmit pattern for a given set of phasor amplitudes for the excitation currents is the same as the receiving pattern with the same phasors used as signal combining coefficients, so for the most part we do not need to distinguish between the transmit and receive cases in developing tools for designing the array configuration or the excitations to achieve a given goal for the radiation or receiving pattern.

The process of determining the radiation pattern in terms of the array element types, locations, and driving currents or excitation coefficients is array analysis. An array antenna design requires a choice of antenna elements, a layout for the positions of the elements, and a beamformer design algorithm that provides

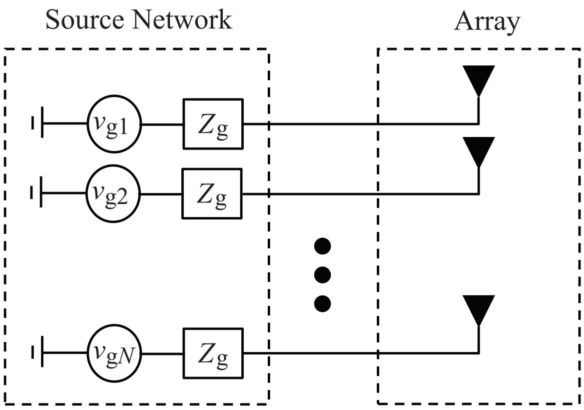

Figure 6.1: Array antenna transmitter block diagram. The array elements are excited by generators with phasor voltages $v_{g1}, v_{g2}, \ldots, v_{gN}$ and source impedance Z_g. The source network may be a corporate feed, transmit modules with phase shifters and possibly variable gain amplifiers, or digitally generated signals with power amplifiers driving each element.

values for the element excitation amplitudes and delays or phases. Designing the elements, locations, and excitations to achieve a desired radiation pattern is array synthesis. Beamformer design or array synthesis and array analysis are closely linked, because an analysis method that gives the pattern in terms of provides important insight into how the beamformer coefficients should be chosen to achieve a particular goal for the radiation pattern.

The theory of array antenna analysis and beamforming can be viewed as a sequence of techniques with increasing generality:

1. 1D linear array of identical, uniformly spaced elements, excitation with equal amplitude currents and a progressive phase shift for beam steering.

2. Linear array with arbitrary excitations, including binomial arrays, Dolph-Chebyshev weights for lowest sidelobes given a main beam width, synthesized patterns, and so forth. The uniform linear array (ULA) falls into this class.

3. Irregularly spaced elements - sparse arrays, thinned arrays, random arrays. The array element locations become additional degrees of freedom that can be used to accomplish various design goals.

4. 2D and 3D arrays - full hemispherical or spherical beam steering.

Most of the qualitative features of the most complex 2D array can be understood by considering 1D arrays, so we will start there.

6.1 Array Factor

From this point forward, we will assume that the array elements are identical and differ only in their locations and driving excitation phase and magnitude. Nearly all arrays employ identical elements, so the treatment is quite general at this point. The major limitation of this assumption is that coupling between elements is neglected. When an element is excited, the radiated fields induce currents on nearby elements, which produces an additional contribution to the radiated fields. The equivalent current for an element in an array has a large component on the driven element, and smaller components on the other elements around the driven elements. For elements near the edge of the array, the coupled currents are different because there are fewer nearby elements than is the case for elements in the interior of the array. Even though the elements in the array are identical, the elements in the array are effectively represented by different equivalent currents. This means that the embedded element radiation pattern (including radiation from currents induced on neighboring elements) is different from the radiation pattern of the antenna element in isolation. In many cases, coupling between elements can be neglected, or if the array is large, edge element effects can be ignored, in which case the analysis given here is an accurate model of the array behavior.

Consider an antenna located at the origin and excited with a unit strength driving current ($I_{\rm in} = 1\,{\rm A}$) with equivalent current $\overline{J}$. The far field is

$$\overline{E}_{el}(\overline{r}) = -j\omega\mu(1-\hat{r}\hat{r})\frac{e^{-jkr}}{4\pi r}\int d\overline{r}'\, e^{j\overline{k}\cdot\overline{r}'}\overline{J}(\overline{r}') \tag{6.1}$$

where $\overline{k} = k\hat{r}$. If the antenna is shifted to the point $\overline{r}_1$, the field becomes

$$\begin{aligned}\overline{E}(\overline{r}) &= -j\omega\mu(1-\hat{r}\hat{r})\frac{e^{-jkr}}{4\pi r}\int d\overline{r}'\, e^{j\overline{k}\cdot\overline{r}'}\overline{J}(\overline{r}'-\overline{r}_1)\\ &= -j\omega\mu(1-\hat{r}\hat{r})\frac{e^{-jkr}}{4\pi r}e^{j\overline{k}\cdot\overline{r}_1}\int d\overline{r}''\, e^{j\overline{k}\cdot\overline{r}''}\overline{J}(\overline{r}'')\\ &= e^{j\overline{k}\cdot\overline{r}_1}\overline{E}_{\rm el}(\overline{r}) \end{aligned}\tag{6.2}$$

This is the Fourier shift theorem, since a shift in the near field location of the antenna led to a phase shift in the far field.

If we have an array of N identical elements driven with phasor excitation currents $I_1, I_2, \ldots, I_N$ and located at the points $\overline{r}_1, \overline{r}_2, \ldots \overline{r}_N$, the total far field can be found by adding the field radiated by each element, scaled so that the input current is $I_{\rm in} = I_n$ rather than $I_{\rm in} = 1A$. This leads to

$$\begin{aligned}\overline{E} &= \sum_{n=1}^{N} I_n e^{j\overline{k}\cdot\overline{r}_n}\overline{E}_{\rm el}(\overline{r})\\ &= \overline{E}_{\rm el}(\overline{r}) \underbrace{\sum_{n=1}^{N} I_n e^{j\overline{k}\cdot\overline{r}_n}}_{\text{Array factor } A(\theta,\phi)}\end{aligned}\tag{6.3}$$

The array factor $A(\theta, \phi)$ can be controlled by adjusting the complex excitations I_n. The array radiation pattern is the product of the array factor and the pattern of one element located at the origin. If the array elements are electrically small, then the element pattern $\overline{E}_{\rm el}$ is broad and slowly varying with angle, and the array factor dominates in determining the shape of the radiation pattern. If the elements were ideal isotropic radiators, then the radiation pattern is equal to the array factor.

The theory of array antenna beamforming provides methods for designing the excitations I_n and locations $\overline{r}_n$ of the elements in order to achieve various goals for the radiation pattern, such as high gain, electronic beam steering, low sidelobes, or interference nulling. Many different approaches to designing array excitations have been developed over the years, ranging from simple analytical formulas to sophisticated numerical optimization routines.

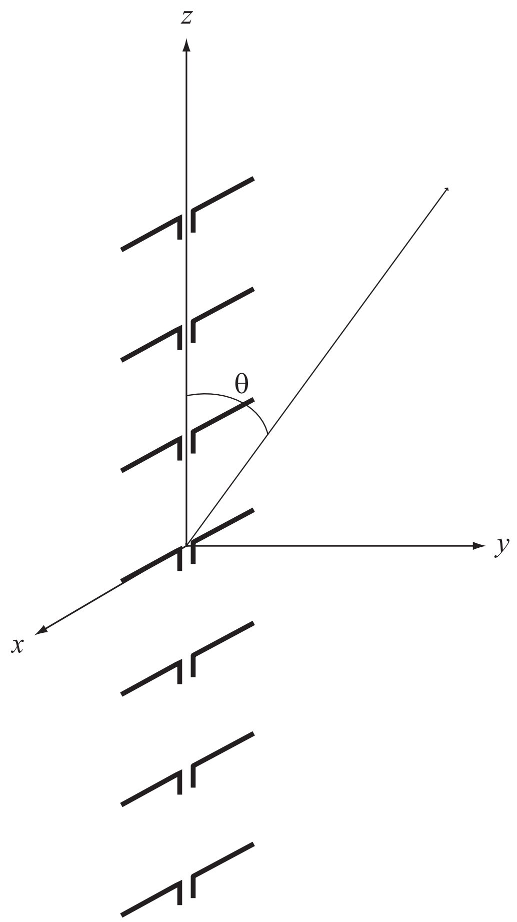

Figure 6.2: Uniform linear array with elements along the z axis.

6.2 The Uniform Linear Array (ULA)

One very common array type consists of elements equally spaced along a line as shown in Figure 6.2. We will place the elements on the z axis symmetrically about $z = 0$, so that

$$\begin{aligned} A(\theta, \phi) &= \sum_{n=-N}^{N} I_n e^{j\bar{k}\cdot nd\hat{z}} \\ &= \sum_{n=-N}^{N} I_n e^{jknd\cos\theta} \end{aligned} \tag{6.4}$$

where the number of elements is $2N + 1$. A similar treatment can be developed for an even number of elements. The radiation pattern is symmetric about the axis of the linear array, so that the pattern is independent of ϕ.

Main lobe. For the case of uniform excitation, $I_n = 1$, the array factor is at a maximum when the $2N + 1$ complex numbers in this summation add in phase. This occurs when $\cos\theta = 0$, or $\theta = \pi/2$. This is a broadside pattern.

Nulls. As θ moves away from $\pi/2$, the phasors in the sum are no longer aligned, but are spaced at different points on the unit circle in the complex plane. The sum decreases, and the radiation pattern also decreases. When θ is such that all the phasors are equally spaced around the unit circle, the sum is zero and the array radiation pattern has a null. This occurs at

$$kd\cos\theta = \frac{2\pi}{2N+1} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{2\pi}{kd(2N+1)}\right) = \cos^{-1}\left(\frac{\lambda}{L+d}\right) \tag{6.5}$$

where $L = 2Nd$ is the total length of the array. If $L+d < \lambda$, then the array factor has no nulls. If $L+d \geq \lambda$, the null to null beamwidth is

$$\psi_{\text{null-null}} = 2\cos^{-1}\left(\frac{\lambda}{L+d}\right) \simeq \frac{2\lambda}{L+d}, \quad L \gg \lambda \tag{6.6}$$

If we further increase θ, the phasors move around the unit circle and the sum is no longer zero, causing a pattern sidelobe. Eventually, if the array is large enough, we arrive at another null. In general, nulls are located at

$$\cos\theta_p = \frac{p}{2N+1}\frac{\lambda}{d}, \quad p = 0, \pm 1, \pm 2, \ldots \tag{6.7}$$

The number of nulls is limited by

$$\frac{p}{2N+1}\frac{\lambda}{d} \leq 1 \quad \Rightarrow \quad p \leq (2N+1)\frac{d}{\lambda} \tag{6.8}$$

Grating lobes. If $kd > 2\pi$, then for θ large enough, $kd\cos\theta = 2\pi$, and we get another angle in addition to $\theta = \pi/2$ where all the phasors in the array factor sum are aligned, causing another pattern lobe with maximum amplitude. These are called grating lobes, and are usually undesirable. For an array with equal phase excitations, to avoid grating lobes the element spacing must be small enough that

$$kd \leq 2\pi \quad \Rightarrow \quad d \leq \lambda \tag{6.9}$$

An array with widely spaced elements has many grating lobes, and the radiation pattern has strong peaks at many angles. Grating lobes can be eliminated by reducing the element spacing or suppressed by choosing an electrically large element with narrow radiation pattern.

From a receiving point of view, if the array is illuminated by a plane wave

$$\overline{E} = \overline{E}_0 e^{-j\overline{k}\cdot\overline{r}} \tag{6.10}$$

then along the z axis where the array is located the field is

$$\overline{E}(0,0,z) = \overline{E}_0 e^{-jkz\cos\theta} \tag{6.11}$$

where θ is the arrival angle of the plane wave at the array relative to the z axis. At the locations of the array elements, the field samples are given by

$$\overline{E}_n = \overline{E}_0 e^{-jknd\cos\theta} \tag{6.12}$$

which for small antenna elements is proportional to the received voltages V_n at the antenna terminals. If the plane wave is incident from the broadside direction ($\theta = \pi/2$), $V_n = 1$. If $d > \lambda$, then at some angle θ away from $\pi/2$, $kd\cos\theta = 2\pi$, and $V_n = 1$ for all n. The samples of the field at the array elements are too widely spaced to discriminate between the two different plane waves.

To avoid grating lobes, the elements must be spaced more closely than one element per wavelength. For arrays with steered beams using progressive phase shifts of the element excitations, we must have $d \leq \lambda/2$, or at least two elements per wavelength, to avoid grating lobes for all beam steering angles. Since the array essentially samples the incident field at the element locations, grating lobes are simply a manifestation of aliasing.

6.2.1 Tapered Excitations

If the magnitudes of the excitation currents I_n are chosen to be tapered across the array instead of equal in magnitude, so that I_0 is largest and $I_{\pm n}$ decrease in magnitude, then θ must move farther from $\pi/2$ to obtain a pattern null than for the uniformly weighted array, and the main lobe is wider. In addition, the phasors add to a smaller value for the sidelobes, and the sidelobe level decreases.

6.2.2 Array Beam Scanning

If the array excitation currents include a progressive phase shift of the form

$$I_n = Ie^{-jn\alpha} \tag{6.13}$$

then the array factor becomes

$$A(\theta) = I\sum_n e^{jn(kd\cos\theta-\alpha)} \tag{6.14}$$

The main beam is now located at

$$kd\cos\theta_0 = \alpha \quad \Rightarrow \quad \theta_0 = \cos^{-1}\left(\frac{\alpha}{kd}\right) \tag{6.15}$$

This expression shows that the main beam angle can be controlled by adjusting the progressive phase shift parameter α.

In addition to changing the main beam angle, beam steering also changes the beam width. The first nulls near the main lobe are at

$$kd\cos\theta_{\pm1} - \alpha = \pm\frac{2\pi}{2N+1} \tag{6.16}$$

The angle between the nulls is

$$\cos\theta_1 - \cos\theta_{-1} = \frac{2\lambda}{L+d} = 2\sin\underbrace{\frac{\theta_1+\theta_{-1}}{2}}_{\simeq\theta_0}\sin\frac{\theta_1-\theta_{-1}}{2} \tag{6.17}$$

so that

$$\sin(\Delta\theta/2) \simeq \frac{\lambda}{L+d}\frac{1}{\sin\theta_0} \tag{6.18}$$

which shows that the main lobe becomes broader as the beam is steered away from broadside and θ_0 moves away from $\pi/2$. Consequently, for a linear array the endfire beam is wider than the broadside beam.

To avoid grating lobes for the steered array, we must have

$$kd\cos\theta + \alpha \le 2\pi \quad \Rightarrow \quad \frac{d}{\lambda} \le \frac{1}{1+|\cos\theta_0|} \le \frac{1}{2} \tag{6.19}$$

In the receiving case, this reduces to the Nyquist sampling criterion for a plane wave arriving at the array from an endfire direction.

To plot the array factor of a ULA by hand, it is convenient to sum the array factor for the uniform amplitude, phase-steered array analytically. Since the array factor for a ULA is a geometrical series, we have that

$$\begin{aligned} A(\theta) &= \sum_{n=-N}^{N} e^{jn(kd\cos\theta-\alpha)} \\ &= \sum_{n=-N}^{N} x^n, \quad x = e^{j(kd\cos\theta-\alpha)} \\ &= x^{-N}\sum_{0}^{2N} x^n \\ &= x^{-N}\frac{1-x^{2N+1}}{1-x} \\ &= \frac{x^{-N-1/2}-x^{N+1/2}}{x^{-1/2}-x^{1/2}} \\ &= \frac{\sin[\frac{1}{2}(2N+1)(kd\cos\theta-\alpha)]}{\sin[\frac{1}{2}(kd\cos\theta-\alpha)]} \end{aligned}$$

This is the Dirichlet function or periodic sinc function. The array factor can be written as

$$F(u) = \frac{\sin[\frac{1}{2}(2N+1)u]}{\sin[\frac{1}{2}u)]} \tag{6.20}$$

where $u = kd\cos\theta - \alpha$. This formula motivates a simple graphical construction to convert from the graph of $F(u)$ in Cartesian coordinates to a polar plot of the array factor, which is referred to as the visible window method.

6.3 Pattern Synthesis

So far, we have looked at array analysis, or procedures for finding the pattern given the array element locations and excitations. Analysis procedures are useful mainly because they provide tools and insight that are helpful for the problem of finding the required excitations to generate a given pattern, or array pattern synthesis. We will now look at several approaches to pattern synthesis. This treatment will be in the context of the ULA, although approaches for a 1D array can often be extended to the 2D case.

6.3.1 Schelkunoff's Unit Circle Representation

The array factor for a ULA can be written as

$$A(\theta) = \sum_{n=0}^{N} I_n e^{jn(kd\cos\theta-\alpha)} \tag{6.21}$$

where we have factored the excitation currents into a progressive phase shift parameterized by α and an additional complex number I_n (i.e., I_n is not the complete current phasor). The progressive phase shift allows beam steering, whereas I_n allows beam shaping.

If we define

$$w = e^{j\psi} = e^{j(kd\cos\theta-\alpha)} \tag{6.22}$$

then the array factor becomes

$$\begin{aligned} A(w) &= \sum_{n=0}^{N} I_n e^{jn\psi} \\ &= \sum_{n=0}^{N} I_n w^n \\ &= I_N \sum_{n=0}^{N} \frac{I_n}{I_N} w^n \\ &= I_N \left(w^N + \frac{I_{N-1}}{I_N} w^{N-1} + \cdots + \frac{I_0}{I_N} \right) \\ &= I_N f(w) \end{aligned} \tag{6.23}$$

Since $f(w)$ is a polynomial with highest order coefficient equal to unity, we can factor f based on its zeros as

$$f(w) = \prod_{n=1}^{N} (w - w_n), \quad w_n = N \text{ roots of } f(w) \tag{6.24}$$

The array factor is determined by the values of this polynomial as w moves around the unit circle in the complex plane.

As the angle θ varies, w may move around the entire unit circle or just a portion, depending on the array element spacing. The phase angle ψ of w lies in the range $\psi_f \le \psi \le \psi_s$, where

$$\begin{aligned} \psi_s &= kd - \alpha \\ \psi_f &= -kd - \alpha \end{aligned}$$

The value of the polynomial is in turn determined by the locations of the N zeros. The fundamental idea is to design f to achieve a given array factor by choosing values for the zeros, and then finding the excitations I_n using (6.23). To do this, we need to develop some intuition into how the array factor relates to the locations of the zeros.

Nulls. If the zeros w_n lie on the unit circle between ψ_s and ψ_f, then the radiation pattern has nulls. To obtain a null-free pattern, we can move the zeros w_n off the unit circle.

Uniform excitation (broadside array). If $I_n = 1$ for all n, then

$$A(w) = f(w) = \sum_{0}^{N} w^n = \frac{1 - w^{N+1}}{1 - w} \tag{6.25}$$

The roots are located at

$$w_n = e^{j2\pi n/(N+1)}, \quad n = 1, 2, \ldots, N \tag{6.26}$$

For $n = 0$, $A(w_0)$ approaches a nonzero limit, so $w = 1$ is not a zero but instead corresponds to a maximum of the pattern. Since the roots are all on the unit circle, the pattern has nulls unless the element spacing is very small.

Scanned arrays. If $\alpha \neq 0$, the polynomial $f(w)$ does not change, but the value of θ corresponding to points on the circle is shifted. This moves the location of the main lobe as a function of the angle θ.

Low sidelobes. From (6.24), the magnitude of the array factor for a given value of w is the product of the distances d_m from w to each of the zeros. If two roots are close together, then the sidelobe height is smaller. For a broadside array, to reduce sidelobes, the roots must cluster near $\psi = \pi$. Moving the roots away from $\psi = 0$ means that the main lobe becomes wide. This reveals a fundamental tradeoff in array design between sidelobe level and main beam width. The Chebyshev array is optimal in the sense that the sidelobes are as low as possible for a given main beam width.

Superdirectivity. If choose the element spacing to be small, so that $kd \to 0$, and change the current excitations so that the zeros of the polynomial $f(w)$ are still located between ψ_s and ψ_f, we can obtain a very large directivity for a very small array. The price for this increase in directivity is that the excitation currents become large in magnitude, so that the ohmic losses in the antenna increase and the gain may not be nearly as large as the directivity.

6.3.2 Directivity of a ULA

As with single antennas, determining the gain of an array antenna requires evaluation of an integral for the total radiated power. For a ULA with half-wavelength element spacing, this integral can be evaluated analytically. For a linear array along the z axis, the directivity pattern is

$$\begin{aligned} D(\theta) &= \frac{4\pi r^2 S(\theta)}{\int_0^{2\pi}\int_0^{\pi} S(\theta) r^2 \sin\theta \, d\theta \, d\phi} \\ &= \frac{2S(\theta)}{\int_0^{\pi} S(\theta)\sin\theta \, d\theta} \end{aligned}$$

If we neglect the element pattern and approximate the elements as isotropic radiators, this becomes

$$D(\theta) \simeq \frac{2|A(\theta)|^2}{\int_0^{\pi} |A(\theta)|^2 \sin\theta \, d\theta}$$

The maximum value of the directivity pattern is

$$D = \frac{2\left|\sum_n I_n\right|^2}{\int_0^{\pi} \sum_m I_m e^{jmu} \sum_n I_n^* e^{-jnu} \sin\theta \, d\theta} \tag{6.27}$$

where $u = kd\cos\theta - \alpha$. With a change of variables from θ to u in the integral, $du = -kd\sin\theta \, d\theta$, and we have

$$D = \frac{2\left|\sum_n I_n\right|^2}{\frac{1}{kd}\int_{-kd-\alpha}^{kd-\alpha} \sum_{m,n} I_m I_n^* e^{j(m-n)u} \, du} \tag{6.28}$$

If kd is an even multiple of π, then the integral vanishes unless $m = n$, and the directivity becomes

$$D = \frac{\left|\sum_n I_n\right|^2}{\sum_n |I_n|^2} \tag{6.29}$$

This expression is a good approximation to the directivity of an array with half-wavelength element spacing and nearly isotropic elements. The ratio is largest when the excitations I_n are equal, so that the uniformly excited array has the largest directivity. This is analogous to the case of an aperture antenna, for which a uniform aperture distribution yields the highest gain.

For other types of arrays, the ratio of the squared magnitude of the sum of the array excitations to the sum of the squares is a poor estimate of directivity. If $kd < \pi$, (6.29) no longer holds, and superdirectivity solutions with higher directivity than the uniformly excited array can be found. Even if it does not accurately estimate the directivity, the ratio in (6.29) still provides a measure of the deviation of the excitations from uniformity. When normalized by $1/N$, so that its peak value is unity, this quantity is sometimes called the taper efficiency.

6.3.3 Dolph-Chebyshev Arrays

A fundamental array design problem is the pencil beam pattern, for which we want the narrowest possible main beam given a maximum sidelobe level. Intuitively, one can argue that for the optimal pattern, all the sidelobes must have the same level, otherwise we could let the lower sidelobe increase slightly and reduce the larger sidelobes, thereby decreasing the overall maximum sidelobe level.

Mathematically, this property is captured by Chebyshev polynomials, which are defined by

$$T_n(x) = \cos[n\cos^{-1}(x)] \tag{6.30}$$

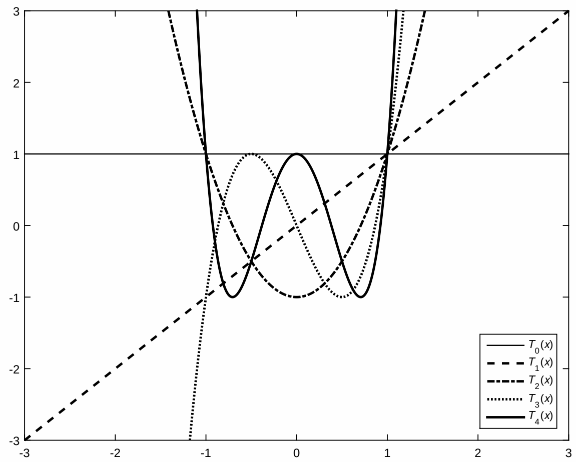

Figure 6.3: Chebyshev polynomials of orders 0, 1, 2, 3, 4. The polynomials have values between -1 and 1 for $-1 \leq x \leq 1$, and the extrema all have magnitude one. This is the equal ripple property of Chebyshev polynomials.

The first few Chebyshev polynomials are

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x
\end{aligned}$$

From the definition (6.30), it can be seen that $|T_n(x)| \leq 1$ for $|x| \leq 1$, and the maxima and minima in this range all reach a magnitude of one. This is the "equal ripple" property. For $n > 1$ and $|x| > 1$, the magnitude goes to infinity, since $\cos^{-1}(x)$ is imaginary. The first few Chebyshev polynomials are shown in Figure 6.3.

To use Chebyshev polynomials as an array factor, we need to transform from the far field angle of the pattern to the argument x of the polynomials. We want $\pi/2$ to correspond so a value of x that is larger than one, in order to form the main lobe. Sidelobe angles should transform to $|x| \leq 1$. This is accomplished by the Dolph transformation

$$x = b\cos(u/2), \quad b > 1 \tag{6.31}$$

where $u = kd\cos\theta - \alpha$. For an N element array, the array factor that we want to obtain is

$$A(u) = T_{N-1}[x(u)] \tag{6.32}$$

where the degree of the Chebyshev polynomial is determined by the fact that an N element array corresponds to a polynomial in u of order $N-1$. In order to realize this radiation pattern, we need to develop a procedure for determining array element excitations that produce an array factor given by (6.32).

With the array elements centered on $z = 0$, the array factor is

$$A(u) = \sum_{n=1}^{N/2} e^{j(n-1/2)u} I_n + \sum_{n=1}^{N/2} e^{-j(n-1/2)u} I_{-n}, \quad (N \text{ even})$$
$$A(u) = I_0 + \sum_{n=1}^{(N-1)/2} e^{jnu} I_n + \sum_{n=1}^{(N-1)/2} e^{-jnu} I_{-n}, \quad (N \text{ odd})$$

We will now assume that the excitations I_n are symmetric about $z = 0$, so that $I_{-n} = I_n$. By combining symmetric terms in the sums, the array factor can be expressed as

$$A(u) = 2\sum_{n=1}^{N/2} \cos[(n-1/2)u] I_n, \quad (N \text{ even}) \tag{6.33a}$$
$$A(u) = I_0 + 2\sum_{n=1}^{(N-1)/2} \cos(nu) I_n, \quad (N \text{ odd}) \tag{6.33b}$$

For the even case, using the Dolph transformation,

$$A(u) = 2\sum_{n=1}^{N/2} \underbrace{\cos[(2n-1)\cos^{-1}(x/b)]}_{T_{2n-1}(x/b)} I_n = T_{N-1}(x) \tag{6.34}$$

so that designing the array requires finding excitations such that

$$2\sum_{n=1}^{N/2} T_{2n-1}(x/b) I_n = T_{N-1}(x) \tag{6.35}$$

A similar result is obtained for the case of N odd. This expression shows that designing the excitations for a Dolph-Chebyshev array requires that we express the order $N-1$ Chebyshev polynomial as a linear combination of the polynomials of order 1, 3, 5, . . ., $N-1$. The coefficients of the linear combination are the excitation currents.

By expanding both sides of (6.34) as polynomials and equating like powers of x, we can find the excitations I_n. Alternately, we can use the fact that the zeros of $T_n(x)$ are given by

$$x_p = \cos\frac{(2p-1)\pi}{2n}, \quad p = 1, 2, \ldots, n \tag{6.36}$$

from which the Schelkunoff unit circle approach can be used to determine the excitations.

It remains to examine the resulting pattern and determine the sidelobe level and main beam width. These pattern properties are controlled by the parameter b. The pattern maximum corresponds to the largest magnitude of the argument of the Chebyshev polynomial, which from (6.31) occurs at $u = 0$ and $x = b$, so that the value of the pattern maximum is

$$A(0) = T_{N-1}(b) \tag{6.37}$$

Since the sidelobes have maxima equal to one, this value is the the square root of the sidelobe level or the ratio of the radiation pattern peak to the largest sidelobe. The first null is located at θ_1, where

$$x_1 = \cos\frac{\pi}{2(N-1)} = b\cos(u/2), \quad u = kd\cos\theta_1 - \alpha \tag{6.38}$$

To design for a given sidelobe level, we use (6.37) to find b. The main beam width is then determined by (6.38). To design for a given main beam width, b is determined from (6.38). The Chebyshev array is a one-parameter array, so that both the sidelobe level and the main beam width are functions of b. If we increase b, the sidelobe level increases, but the main beam becomes wider. If we decrease b, the main beam becomes narrower, but the sidelobes increase. Since the Chebyshev array is the optimal solution for minimum main beam width given a sidelobe level, this represents a fundamental tradeoff in array antenna design.

6.3.4 Other Pattern Synthesis Methods

Taylor arrays. One problem with the Chebyshev array is that for large numbers of elements, the excitations for elements near the ends of the array must become large in order to maintain equal amplitude sidelobes far from the main beam. Taylor's solution to this problem was to sacrifice a small increase in the main beam width in order to allow the sidelobes to fall off more naturally in the far field. The Taylor pattern is a compromise between the Chebyshev pattern and the sinc-like pattern of a uniformly excited array. The weights can be designed analytically by sampling a continuous aperture distribution.

Orchard's method. Using Schelkunoff's representation, with the array factor in polynomial form, the derivatives of the pattern with respect to the zeros can be computed analytically, which allows the locations of the zeros to be updated numerically based on the difference between an initial pattern and the desired pattern. This provides an efficient iterative procedure for obtaining a specified pattern.

Fourier series. If $kd = \pi$, then the array factor

$$A(u) = \sum_n I_n e^{jnu} \tag{6.39}$$

becomes a Fourier series, and the excitations can be found from the Fourier series expansion of the desired radiation pattern.

Other methods. Trigonometric interpolation, adaptive arrays based on measured signal and noise responses, null steering, and brute force optimization are other approaches to obtaining specified radiation patterns or patterns with desired properties.