Chapter 12

Multiple Input, Multiple Output (MIMO) Communications

Diversity uses multiple antennas to increase the performance of a channel, but the end result of the basic diversity techniques surveyed in the previous chapter is a single input, single output (SISO) channel with greater SNR. More generally, we can transmit different signals from each antenna and process the received signals using multiple detectors to achieve several effective parallel communication channels. Whereas higher SNR improves capacity only logarithmically, multiple parallel channels increases the achievable information rate multiplicatively. This is the motivation for a multiple input, multiple output (MIMO) communication channel.

We will first derive a bound on the capacity of a MIMO channel in terms of the properties of the propagation environment, and then explore specific channel types and methods for exploiting a MIMO channel to achieve capacity that approaches the theoretical limit. We will find that the performance of MIMO is related to the multipath richness of the environment. Being a theoretical framework for antenna diversity systems, MIMO will also be shown to be related to the combining methods of the previous chapter. MIMO also can be considered within a coding theory framework, and the spatial symbols that are transmitted at each array element can be linked to the temporal codes used for channel coding, leading to the concept of space-time codes.

12.1 MIMO Channel Model

For a multiple input, multiple output communication system, both the transmitter and receiver consist of antenna arrays. We will denote the number of transmit antennas as $N_t$ and the number of receive antennas as $N_r$. The transmitted signals will be characterized in terms of the phasor or complex baseband representation of generator open circuit voltages attached to the input terminals of the transmit antennas. We will arrange these values into the column vector

$$s = \begin{bmatrix} s_1(n) \\ s_2(n) \\ \vdots \\ s_{N_t}(n) \end{bmatrix} \quad (12.1)$$
Each element is a complex voltage representing the symbol transmitted at the \( n \)th symbol period. At the receiver, we define

\[
x = \begin{bmatrix}
x_1(n) \\
x_2(n) \\
\vdots \\
x_{N_r}(n)
\end{bmatrix}
\]

(12.2)

to be a vector of voltages at the outputs of complex baseband receivers connected to each antenna.

In terms of these quantities, we can represent the propagation channel in matrix form,

\[
x(n) = H(n)s(n) + w(n)
\]

(12.3)

where \( H \) relates input currents or voltages at the transmit antenna ports to received voltages and \( w \) is a random process representing external noise, interference, noise due to ohmic losses in the receiving antenna elements, and receiver noise. The noise vector \( w \) in this chapter is not to be confused with the beamformer weight vector in earlier chapters. The index \( n \) represents a given symbol period over time.

For a fixed channel, we can view \( H \) as a deterministic matrix with specified values, or for a variable channel we can model the elements of \( H \) as random processes. The transmitted symbols \( s \) can also be modeled as random processes, since the details of a specific data stream sent across the channel is not important in designing the communications systems. In the following, we will not consider dispersion and time-delay effects. Dispersion and time delay nearly always occur in multipath propagation environments, but we will assume that these effects can be dealt with and removed using signal processing.

### 12.1.1 Network Model for a MIMO Channel

To further understand the channel matrix, we can model a combined transceiver system as a multiport network with \( N_1 \) ports at the transmit side and \( N_2 \) ports at the receiver. The mutual impedance matrix can be written in block form as

\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
\]

(12.4)

\( Z_{11} \) is the mutual impedance matrix of array 1 and \( Z_{22} \) is the mutual impedance matrix of array 2. If array 1 is the transmitter, then the propagation channel is represented by \( Z_{21} \). This matrix relates the vector of input currents \( i_1 \) into the ports of array 1 to the open circuit voltages \( v_2 \) induced at the ports of array 2. In terms of the mutual impedances, the channel matrix is

\[
H = Q_2Z_{21}Q_1
\]

(12.5)

where \( Q_1 \) is the system transfer function matrix on the right side of the transmit network relationship (7.6) and \( Q_2 \) is defined similarly by (8.5) for array 2 as the receiver. \( Q_1 \) relates generator voltages to the transmit element input currents and \( Q_2 \) relates the receive antenna open circuit voltages to receiver output voltages. The properties of the propagation environment are characterized by \( Z_{21} \). If the details of the transmitter and receiver electronics are not important, we could redefine \( s \) and \( x \) in (12.3) to be the transmit element input currents and the receiver element open circuit voltages, respectively, and represent the channel directly as \( H = Z_{21} \).

### 12.1.2 Free Space Channel

Using the array formulation of Chapter 6, the open circuit voltage at the \( n \)th element of array 2 due to radiated fields from array 1 is

\[
v_{oc,n} = \frac{4\pi jr_2 e^{jk_2}}{\omega \mu I_1 I_2} \sum_{m=1}^{N_t} i_m E_m^1(\tau_1) \cdot E_n^2(\tau_2)
\]

(12.6)

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where $E_{1}^{n}$ is the embedded element radiation field pattern with current $I_{1}$ into element $n$ of array 1 and the other elements open circuited. $E_{2}^{n}$ is defined similarly for array 2. From this result, we can see that

$$Z_{21mn} = \frac{4\pi r_{2}e^{jkr_{2}}}{\omega \mu I_{1}I_{2}} E_{m}^{1}(r_{1}) \cdot E_{n}^{2}(r_{2})$$

(12.7)

The point $r_{1}$ is the location of array 2 in the coordinate system used to find the embedded element patterns of array 1. The point $r_{2}$ has an arbitrary length $r_{2}$ and points in the direction of array 1 in the coordinate system of array 2. The decrease in signal strength with the separation between transmitter and receiver is represented by the $r_{1}$ dependence of the fields $E_{m}^{1}(r_{1})$ radiated by array 1.

In earlier chapters, the embedded element patterns $E_{1}^{n}$ and $E_{2}^{n}$ were defined with the arrays in free space. Equation 12.7 could be modified to accommodate the actual environment by redefining $E_{1}^{n}$ to be the radiation pattern of the $n$th element of the transmit array in the presence of the multipath propagation environment. Another way to include multipath is to include an angular spectrum transfer function that relates a plane wave propagation direction away from the transmitter to an arrival direction at the receiver, along the lines of (9.40).

### 12.2 Capacity of the Gaussian MIMO Channel

In order to analyze the capacity of a MIMO system, we need to generalize the Shannon capacity in (10.52) to a multiple input, multiple output channel. The simplest approach would be to consider each transmitter and receiver pair as an independent channel, and multiply the SISO capacity bound by the smaller of $N_{t}$ and $N_{r}$. The problem with this approach is that the received signals are a combination of all the transmitted signals, and the capacity can be increased by using the channels in a cooperative manner. We must take a more sophisticated approach to understand the actual capacity of a MIMO channel.

In the analysis of the capacity of a MIMO channel, we must maximize the mutual information (10.50) between the transmit symbol vector and the receiver outputs over all possible distributions for the signal symbols $x$. The goal when designing a MIMO system is to select transmit symbol vectors $s$ somehow to achieve a bit rate as close to the capacity as possible. If we chose a given modulation scheme such as QPSK, the resulting mutual information and bit rate would not be maximum, since a richer symbol constellation with more symbols could better exploit the propagation channel and increase the achievable bit rate. If we model $s$ as a random process, we have essentially two degrees of freedom: the PDF of each element of $s$ and the correlation matrix $R_{s}$. It can be shown that for an AWGN channel, which has Gaussian distributed noise, the Gaussian distribution for the signal maximizes capacity. In practice, the actual transmitted symbol distribution is not Gaussian, so the realized bit rate does not reach the theoretical capacity that we are deriving here, but the capacity bound will allow us to assess the relative goodness of various types of channels and quantify how close a given modulation scheme comes to achieving the optimum.

Based on this argument, the transmitted symbols at each time step $n$ in this analysis will be modeled as a random vector with zero mean complex Gaussian distribution. The covariance matrix of the transmit symbol vector,

$$R_{s} = E[ss^{H}]$$

(12.8)

must also be specified to fully define the Gaussian random vector. The covariance matrix will not be given at this stage of the derivation, but will be treated in Section 12.4. We will also assume that the symbol vector, channel, and noise are all independent.

The capacity is defined in terms of the mutual information as

$$C = \max_{p(s)} E_{H}[I(s;x|H)]$$

(12.9)

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This expression assumes a fixed realization of the channel matrix $H$. If the propagation channel is modeled stochastically, we can find the average capacity by taking an expectation over the channel matrix, but for now, we will treat the channel matrix as deterministic. From the definition of mutual information,

$$I(s; x|H) = \int \int f_{s,x|H}(s,x|H) \log_2 \left( \frac{f_{s,x|H}(s,x|H)}{f_x(s)f_s(s)} \right) ds dx$$

$$= H(x|H) - H(x)$$

$$= H(x|H) - H(Hs+w|x)$$

$$= H(x|H) - H(w) \quad (12.10)$$

where the last equality holds because only the noise makes the received signal vector $x$ uncertain given a transmitted signal and channel matrix.

This derivation reduces the problem to the determination of the entropy of a zero mean Gaussian random vector. For a real, zero mean Gaussian random vector, the entropy is

$$H(x) = -E[\log_2 f_x(x)]$$

$$= -\int \frac{e^{-x^T R_x^{-1} x/2}}{(2\pi)^N \sqrt{|R_x|}} f_x(x) dx$$

$$= -\int f(x) \frac{1}{2} \left[ \log_2 (2\pi)^N - \log_2 |R_x| - x^T R_x^{-1} x \log_2 (e) \right] dx$$

$$= \frac{1}{2} \left[ N \log_2 2\pi + \log_2 |R_x| + N \log_2 (e) \right] \int f(x) dx$$

$$= \frac{1}{2} \left[ N \log_2 2\pi + \log_2 |R_x| + N \log_2 (e) \right] \quad (12.11)$$

since $E[x^H R_x^{-1} x] = \sum_{ij} R_{ij}^{-1} R_{ij} = N$. For the circular complex case, the entropy doubles and the factor of one half is removed.

Since the transmitted symbols are complex Gaussian, we can also assume that the received signal vector is also complex Gaussian, and (12.11) can be used to obtain the entropy of $s$ as well as the noise vector $w$, so that

$$H(x|H) = N \log_2 2\pi + \log_2 |R_x| + N \log_2 (e)$$

$$H(w) = N \log_2 2\pi + \log_2 |R_w| + N \log_2 (e)$$

The covariance matrix of the received signal in terms of the transmitted symbols and noise is

$$R_x = E[x^H x]$$

$$= E[(Hs+w)^H (Hs+w)]$$

$$= E[Hss^H H^H] + E[ww^H]$$

$$= H E[ss^H] H^H + E[ww^H]$$

$$= HR_a H^H + R_w \quad (12.12)$$

For a given channel matrix, the capacity is

$$C = \max_{R_a} [N \log_2 (e) + N \log_2 2\pi + \log_2 |R_x| - (N \log_2 (e) + N \log_2 2\pi + \log_2 |R_w|)]$$

$$= \max_{R_a} \log_2 \frac{|R_x|}{|R_w|}$$

$$= \max_{R_a} \log_2 \frac{|R_w + HR_a H^H|}{|R_w|} \quad (12.13)$$
The noise covariance matrix must be invertible, or else there is effectively a zero noise channel with infinite capacity. Using the properties of the determinant,

\[ C = \max_{\mathbf{R}_s} \log_2 \det (\mathbf{I}_{N_t} + \mathbf{R}_w^{-1} \mathbf{H} \mathbf{R}_s \mathbf{H}^H) \]  

(12.14)

This is the “log-det” time average or ergodic capacity bound for a MIMO channel. If the channel is stochastic, the capacity is the expectation of this expression over the channel distribution. Intuitively, the second term inside the determinant is similar to an SNR, because it is the product of the received signal covariance matrix and the inverse of the noise covariance matrix.

### 12.2.1 Power Constraint

The channel capacity increases with the magnitude of \( s \), which is determined by the strength of the transmitted signals. In order to obtain a finite capacity when maximizing (12.14) over \( \mathbf{R}_s \), we must limit the time average transmitted power, which is

\[ P_t = \frac{1}{2} E \left[ \text{Re} \{ s^H \mathbf{Q}_1^H \mathbf{Z}_{11} \mathbf{Q}_1 s \} \right] \]  

(12.15)

Since the real part of the impedance matrix \( \mathbf{Z}_{11} \) is determined by the element pattern overlap matrix, this can also be expressed using the overlap matrix according to (7.12).

If the transmit array is uncoupled and the elements are identical and resonant (so that the input reactance is zero), then \( \text{Re} \{ \mathbf{Z}_{11} \} = R_{\text{rad}} \mathbf{I} \). If the elements are conjugate matched to the generators and loads, then \( \mathbf{Q}_1 = \mathbf{Q}_2 = \frac{1}{2} \mathbf{I} \). In this case, the transmitted power becomes

\[ P_t = \frac{1}{8} R_{\text{rad}} E[|s|^4] = \frac{1}{8} R_{\text{rad}} \text{tr} \mathbf{R}_s \]  

(12.16)

where \( \text{tr} \) denotes the matrix trace. So, in the uncoupled case we can express the power constraint in the form

\[ \max_{\mathbf{R}_s} \text{tr} \mathbf{R}_s \leq P_t \]  

(12.17)

where we have lumped the scale factors into \( P_t \). This is a trace constraint on the signal correlation matrix. If the array elements are mutually coupled, the trace constraint is only an approximation to the actual transmitted power.

### 12.3 Special Cases

In order to gain insight into the capacity expression for a Gaussian MIMO channel, we will consider a few special cases for which the capacity bound can be put into a simpler closed form.

#### 12.3.1 Single Input Single Output (SISO) Channel

If \( N_r = N_t = 1 \), then the channel matrix has only one element \( H_{11} \), and the capacity reduces to

\[ C = \log_2 \left[ \frac{\det (\sigma_w^2 + H_{11}^2 \sigma_s^2)}{\det (\sigma_w^2)} \right] \]

\[ = \log_2 \left( \frac{\sigma_w^2 + |H_{11}|^2 \sigma_s^2}{\sigma_w^2} \right) \]

\[ = \log_2 \left( 1 + \frac{|H_{11}|^2 \sigma_s^2}{\sigma_w^2} \right) \]

where \( \sigma_s^2 \) is the transmitted power and \( \sigma_w^2 \) is the noise power at the receiver load. The second term inside the logarithm is the SNR at the receiver, so this expression is the Shannon capacity (10.52) expressed in bits/sec/Hz for a single complex channel.
12.3.2 Receive Diversity

In this case, \( N_t = 1 \), and the capacity is

\[
C = \log_2 \left\{ \frac{\det(\sigma_n^2 I + H\sigma_n^2 H^H)}{\det(\sigma_n^2 I)} \right\}
= \log_2 \det \left( I + \frac{\sigma_n^2}{\sigma_w^2} HH^H \right)
\]

Because the channel matrix is a column vector, the rank-one matrix \( HH^H \) has one eigenvalue equal to \( H^H H \) and the other eigenvalues are zero. Consequently, the eigenvalues of the matrix under the determinant operation are all unity except for one which is equal to \( 1 + H^H H \sigma_n^2 / \sigma_w^2 \). Since the determinant is equal to the product of the eigenvalues, the capacity is

\[
C = \log_2 \left( 1 + \frac{\sigma_n^2}{\sigma_w^2} \right)
= \log_2 \left( 1 + \frac{\sigma_n^2}{\sigma_w^2} \sum_{n=1}^{N_t} |H_{n1}|^2 \right)
\]

(12.18)

The quantity \( \sum_n |H_{n1}|^2 \sigma_n^2 / \sigma_w^2 \) is the SNR that is obtained with maximum ratio combining, as should be expected since maximum ratio combining maximizes the SNR and hence also the capacity of a channel with receive diversity.

12.3.3 Transmit Diversity

If \( N_t = 1 \), the capacity reduces to

\[
C = \log_2 \left( 1 + \frac{\sigma_n^2}{\sigma_w^2} \sum_{n=1}^{N_t} |H_{1n}|^2 \right)
\]

(12.19)

which is nearly identical to (12.18), except that the transmit power in the uncoupled approximation is \( P_t = \text{tr} R_s = \sigma_n^2 N_t \). If we constrain the total transmit power for systems with transmit and receive diversity to be the same, the SNR for transmit diversity is lower by a factor of \( N_t \).

12.3.4 Unknown Channel

If the channel matrix \( H \) is unknown, then we have no way to choose the transmit symbol vector \( s \) to maximize capacity. In this case, one can do no better than to choose transmit symbols such that the elements of \( s \) are uncorrelated, so that

\[
R_s = \sigma_n^2 I
\]

(12.20)

The capacity is

\[
C = \log_2 \det \left( I + \frac{\sigma_n^2}{\sigma_w^2} HH^H \right)
\]

(12.21)

If we set \( P_t = \text{tr} R_s = N_t \sigma_n^2 \), then this becomes

\[
C = \log_2 \det \left( I + \frac{P_t}{N_t \sigma_w^2} HH^H \right)
\]

(12.22)